

EXTERNAL FIELD QED ON CAUCHY SURFACES FOR VARYING ELECTROMAGNETIC FIELDS

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October 15, 2015

Abstract

The Shale-Stinespring Theorem (1965) together with Ruijsenaar's criterion (1977) provide a necessary and sufficient condition for the implementability of the evolution of external field quantum electrodynamics between constant-time hyperplanes on standard Fock space. The assertion states that an implementation is possible if and only if the spatial components of the external electromagnetic four-vector potential A_μ are zero. We generalize this result to smooth, space-like Cauchy surfaces and, for general A_μ , show how the second-quantized Dirac evolution can always be implemented as a map between varying Fock spaces. Furthermore, we give equivalence classes of polarizations, including an explicit representative, that give rise to those admissible Fock spaces. We prove that the polarization classes only depend on the tangential components of A_μ w.r.t. the particular Cauchy surface, and show that they behave naturally under Lorentz and gauge transformations.

1 Introduction and Setup

We consider the external field model of quantum electrodynamics (QED) or no-photon QED which describes a Dirac sea of electrons evolving subject to a prescribed external electromagnetic four-vector potential A_μ . To infer the evolution operator of this model one attempts to implement the one-particle Dirac evolution

$$(i\cancel{\partial} - \cancel{A})\psi = m\psi \quad (1)$$

in second-quantized form. Here, $m > 0$ denotes the mass of the electron; the elementary charge of the electron e (having a negative sign in the case of an electron) is already absorbed in A ; units are chosen such that $\hbar = 1$ and $c = 1$. The employed relativistic notation is introduced with all other notations in Section 1.3. For sake of simplicity we will restrict us to smooth and compactly supported A_μ , i.e.,

$$A = (A_\mu)_{\mu=0,1,2,3} = (A_0, \mathbf{A}) \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4), \quad (2)$$

although this condition is unnecessarily strong.

It is well-known [21, 18] that, on standard Fock space and for equal-time hyperplanes, a second quantization of the one particle Dirac evolution (1) is possible if and only if $\mathbf{A} = 0$, i.e., the spatial components of the external field vanish – a condition that appears strange in view of gauge invariance. In physics the ill-definedness of the evolution operator and its generator for general vector potentials A is usually ignored at first which later manifests itself in the appearance of infinities in informal perturbation series. Those infinities have to be taken out by hand or, as for example in the case of the vacuum expectation value of the charge current, absorbed in the coefficient of the electron charge. Nevertheless, since the sole interaction arises only from a prescribed four-vector field one may rather expect that it should be possible to control the time evolution non-perturbatively. One way to construct a well-defined second-quantized time evolution operator, as sketched in [6], is to implement it between time-varying Fock spaces. Such constructions have been carried out, e.g., in [14, 15, 2]. While the idea of changing Fock spaces might be unfamiliar as seen from the non-relativistic setting, in a relativistic formulation it is to be expected. A Lorentz boost for instance may tilt an equal-time hyperplane to a space-like hyperplane Σ , which requires a change from standard Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ to one attached to Σ , and likewise, for the corresponding Fock spaces.

In this work we extend the existing constructions in [14, 15, 2], which deal exclusively with equal-time hyperplanes, by implementing the second-quantized Dirac evolution from one Cauchy surface to another. The resulting formulation of external field QED has several advantages: 1) Its Lorentz and gauge covariance can be made explicit; 2) as it treats the initial value problem for general Cauchy surfaces it allows to study the evolution in the form of local deformations of Cauchy surfaces in the spirit of Tomonaga and Schwinger, e.g., [22, 20]; 3) it gives a geometric and more general version of the implementability condition $\mathbf{A} = 0$ that was found in the special case of equal-time hyperplanes.

Before presenting our main results in Section 1.1 we outline the construction of the evolution operator for general space-like Cauchy surfaces. Given a Cauchy surface Σ in Minkowski space-time (see Definition 1.9 below), the states of the Dirac sea on Σ are represented by vectors in a conveniently chosen Fock space, here, denoted by the symbol $\mathcal{F}(V, \mathcal{H}_\Sigma)$. In this notation \mathcal{H}_Σ is the Hilbert space of \mathbb{C}^4 -valued, square integrable functions on Σ (see Definition 1.10 below) and $V \in \text{Pol}(\mathcal{H}_\Sigma)$ is one of its polarizations:

Definition 1.1. *Let $\text{Pol}(\mathcal{H}_\Sigma)$ denote the set of all closed, linear subspaces $V \subset \mathcal{H}_\Sigma$ such that V and V^\perp are both infinite dimensional. Any $V \in \text{Pol}(\mathcal{H}_\Sigma)$ is called a polarization of \mathcal{H}_Σ . For $V \in \text{Pol}(\mathcal{H}_\Sigma)$, let $P_\Sigma^V : \mathcal{H}_\Sigma \rightarrow V$ denote the orthogonal projection of \mathcal{H}_Σ onto V .*

The Fock space corresponding to polarization V on Cauchy surface Σ is then defined by

$$\mathcal{F}(V, \mathcal{H}_\Sigma) := \bigoplus_{c \in \mathbb{Z}} \mathcal{F}_c(V, \mathcal{H}_\Sigma), \quad \mathcal{F}_c(V, \mathcal{H}_\Sigma) := \bigoplus_{\substack{n, m \in \mathbb{N}_0 \\ c = m - n}} (V^\perp)^{\wedge n} \otimes \overline{V}^{\wedge m}, \quad (3)$$

where \bigoplus denotes the Hilbert space direct sum, \wedge the antisymmetric tensor product of Hilbert spaces, and \overline{V} denotes the conjugate complex vector space of V , which coincides with V as a set and has the same vector space operations as V with the exception of the scalar multiplication, which is redefined by $(z, \psi) \mapsto z^* \psi$ for $z \in \mathbb{C}$, $\psi \in V$.

Each polarization V splits the Hilbert space \mathcal{H}_Σ into a direct sum, i.e., $\mathcal{H}_\Sigma = V^\perp \oplus V$. The so-called standard polarizations \mathcal{H}_Σ^+ and \mathcal{H}_Σ^- are determined by the orthogonal projectors P_Σ^+ and P_Σ^- onto the free positive and negative energy Dirac solutions, respectively, restricted to Σ :

$$\mathcal{H}_\Sigma^+ := P_\Sigma^+ \mathcal{H}_\Sigma = (1 - P_\Sigma^-) \mathcal{H}_\Sigma, \quad \mathcal{H}_\Sigma^- := P_\Sigma^- \mathcal{H}_\Sigma. \quad (4)$$

Loosely speaking, in terms of Dirac's hole theory, the polarization $V \in \text{Pol}(\mathcal{H}_\Sigma)$ indicates the “sea level” of the Dirac sea, and electron wave functions in V^\perp and V are considered to be “above” and “below” sea level, respectively. However, it has to be stressed that the mathematical structure of the external field problem in QED does not seem to discriminate between particular choices of polarizations V . Hence, unless an additional physical condition is delivered, the V -dependent labels “electron” and “positron” are somewhat arbitrary, and V should rather be regarded as a choice of coordinate system w.r.t. which the states of the Dirac sea are represented. To describe pair-creation on the other hand it is necessary to have a distinguished V , and the common (and seemingly most natural) ad hoc choice in situations when the external field vanishes is $V = \mathcal{H}_\Sigma^-$. Nevertheless, it is conceivable that only a yet to be found full version of QED, including the interaction with the photon field, may distinguish particular polarizations V in general situations.

Given two Cauchy surfaces Σ, Σ' and two polarizations $V \in \text{Pol}(\mathcal{H}_\Sigma)$ and $W \in \text{Pol}(\mathcal{H}_{\Sigma'})$ a sensible lift of the one-particle Dirac evolution $U_{\Sigma'\Sigma}^A : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\Sigma'}$ (see Definition 1.13) should be given by a unitary operator $\tilde{U}_{\Sigma'\Sigma}^A : \mathcal{F}(V, \mathcal{H}_\Sigma) \rightarrow \mathcal{F}(W, \mathcal{H}_{\Sigma'})$ that fulfills

$$\tilde{U}_{\Sigma'\Sigma}^A \psi_{V,\Sigma}(f) (\tilde{U}_{\Sigma'\Sigma}^A)^{-1} = \psi_{W,\Sigma'}(U_{\Sigma'\Sigma}^A f), \quad \forall f \in \mathcal{H}_\Sigma. \quad (5)$$

Here, $\psi_{V,\Sigma}$ denotes the Dirac field operator corresponding to Fock space $\mathcal{F}(V, \Sigma)$, i.e.,

$$\psi_{V,\Sigma}(f) := b_\Sigma(P_\Sigma^{V^\perp} f) + d_\Sigma^*(P_\Sigma^V f), \quad \forall f \in \mathcal{H}_\Sigma. \quad (6)$$

Here, b_Σ, d_Σ^* denote the annihilation and creation operators on the V^\perp and \overline{V} sectors of $\mathcal{F}_c(V, \mathcal{H}_\Sigma)$, respectively. Note that $P_\Sigma^V : \mathcal{H} \rightarrow \overline{V}$ is *anti-linear*; thus, $\psi_{V,\Sigma}(f)$ is anti-linear in its argument f . The condition under which such a lift $\tilde{U}_{\Sigma'\Sigma}^A$ exists can be inferred from a straight-forward application of Shale and Stinespring's well-known theorem [21]:

Theorem 1.2 (Shale-Stinespring). *The following statements are equivalent:*

- (a) *There is a unitary operator $\tilde{U}_{\Sigma'\Sigma}^A : \mathcal{F}(V, \mathcal{H}_\Sigma) \rightarrow \mathcal{F}(W, \mathcal{H}_{\Sigma'})$ which fulfills (5).*
- (b) *The off-diagonals $P_{\Sigma'}^{W^\perp} U_{\Sigma'\Sigma}^A P_\Sigma^V$ and $P_{\Sigma'}^W U_{\Sigma'\Sigma}^A P_\Sigma^{V^\perp}$ are Hilbert-Schmidt operators.*

Note that the phase of the lift is not fixed by condition (5). Even worse, as indicated earlier, depending on the external field A this condition is not always satisfied; see [18]. On the other hand, the choices made for the polarizations V and W were completely arbitrary. We shall see next that adapting these choices carefully will however yield an evolution of the Dirac sea in the corresponding Fock space representations.

There is a trivial but not so useful choice. Pick a Σ_{in} in the remote past of the support of A fulfilling

$$\Sigma_{\text{in}} \text{ is a Cauchy surface such that } \text{supp } A \cap \Sigma_{\text{in}} = \emptyset. \quad (7)$$

Then the choices $V = U_{\Sigma\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^-$ and $W = U_{\Sigma'\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^-$ fulfill (b) of Theorem 1.2 as the off-diagonals are zero. The drawback of these choices is that the resulting lift depends on the whole history of A between Σ_{in} and Σ, Σ' . Moreover, such V and W are rather implicit. But statement (b) in Theorem 1.2 also allows to differ from the projectors P_{Σ}^V and $P_{\Sigma'}^W$ by a Hilbert-Schmidt operator. Hence, it lies near to characterize polarizations according to the following classes:

Definition 1.3 (Physical Polarization Classes). *For a Cauchy surface Σ we define*

$$\mathcal{C}_{\Sigma}(A) := [U_{\Sigma\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^-]_{\approx}, \quad (8)$$

where for $V, V' \in \text{Pol}(\mathcal{H}_{\Sigma})$, $V \approx V'$ means that $P_{\Sigma}^V - P_{\Sigma}^{V'} \in I_2(\mathcal{H}_{\Sigma})$, i.e., is a Hilbert-Schmidt operator $\mathcal{H}_{\Sigma} \hookrightarrow \mathcal{H}_{\Sigma}$.

The equivalence relation \approx can be refined to give another equivalence relation \approx_0 describing polarization classes of equal charge; c.f. [2] and Remark 1.8. As a simple corollary of Theorem 1.2 one gets:

Corollary 1.4 (Dirac Sea Evolution). *Let Σ, Σ' be Cauchy surfaces. Then any choice $V \in \mathcal{C}_{\Sigma}(A)$ and $W \in \mathcal{C}_{\Sigma'}(A)$ implies condition (b) of Theorem 1.2 and therefore the existence of a lift $\tilde{U}_{\Sigma\Sigma'}^A : \mathcal{F}(V, \mathcal{H}_{\Sigma}) \rightarrow \mathcal{F}(W, \mathcal{H}_{\Sigma'})$ obeying (5).*

Consequently, any choice $V \in \mathcal{C}_{\Sigma}(A)$ and $W \in \mathcal{C}_{\Sigma'}(A)$ gives rise to a lift of the one-particle Dirac evolution between the corresponding $\mathcal{F}(V, \mathcal{H}_{\Sigma})$ and $\mathcal{F}(W, \mathcal{H}_{\Sigma'})$ that is unique up to a phase. The crucial questions are: 1) On which properties of A and Σ do these polarization classes depend? 2) How do they behave under Lorentz and gauge transforms? 3) Is there an explicit representative for each class? These question will be answered by our main results given in the next section. The next important question is about the unidentified phase. Although transition probabilities are independent of this phase, dynamic quantities like the charge current will depend directly on it. We briefly discuss this in Section 1.2 and give an outlook of what needs to be done to derive the vacuum expectation of the polarization current.

1.1 Main Results

The definition (8) of the physical polarization classes involves the one-particle Dirac evolution operator and is therefore not very useful in finding an explicit description of admissible Fock spaces for the implementation of the second-quantized Dirac evolution. In our main results Theorems 1.5-1.7 we give a more direct identification of the polarization classes $\mathcal{C}_{\Sigma}(A)$ and state some of their fundamental geometric properties.

The first one ensures that the classes $\mathcal{C}_{\Sigma}(A)$ are independent of the history of A , instead they depend on the tangential components of A on Σ only.

Theorem 1.5 (Identification of Polarization Classes). *Let Σ be a Cauchy surface and let A and \tilde{A} be two smooth and compactly supported external fields. Then*

$$\mathcal{C}_{\Sigma}(A) = \mathcal{C}_{\Sigma}(\tilde{A}) \quad \Leftrightarrow \quad A|_{T\Sigma} = \tilde{A}|_{T\Sigma} \quad (9)$$

where $A|_{T\Sigma} = \tilde{A}|_{T\Sigma}$ means that for all $x \in \Sigma$ and $y \in T_x\Sigma$ we have $A_{\mu}(x)y^{\mu} = \tilde{A}_{\mu}(x)y^{\mu}$.

Ruijsenaar's result, see [18], may be viewed as the special case of this theorem pertaining to $\tilde{A} = 0$ and, for t fixed, $\Sigma = \Sigma_t = \{x \in \mathbb{R}^4 \mid x^0 = t\}$ being an equal-time hyperplane.

Furthermore, the polarization classes transform naturally under Lorentz and gauge transformations:

Theorem 1.6 (Lorentz and Gauge Transforms). *Let $V \in \text{Pol}(\mathcal{H}_\Sigma)$ be a polarization.*

- (i) *Consider a Lorentz transformation given by $L_\Sigma^{(S,\Lambda)} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\Lambda\Sigma}$ for a spinor transformation matrix $S \in \mathbb{C}^{4 \times 4}$ and an associated proper orthochronous Lorentz transformation matrix $\Lambda \in \text{SO}^\uparrow(1, 3)$, cf. [3, Section 2.3]. Then:*

$$V \in \mathcal{C}_\Sigma(A) \quad \Leftrightarrow \quad L_\Sigma^{(S,\Lambda)} V \in \mathcal{C}_{\Lambda\Sigma}(\Lambda A(\Lambda^{-1} \cdot)). \quad (10)$$

- (ii) *Consider a gauge transformation $A \mapsto A + \partial\Omega$ for some $\Omega \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$ given by the multiplication operator $e^{-i\Omega} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_\Sigma$, $\psi \mapsto \psi' = e^{-i\Omega}\psi$. Then:*

$$V \in \mathcal{C}_\Sigma(A) \quad \Leftrightarrow \quad e^{-i\Omega} V \in \mathcal{C}_\Sigma(A + \partial\Omega). \quad (11)$$

As we are mainly interested in a *local* study of the second-quantized Dirac evolution, we only allow compactly supported vector potentials A , and therefore, have to restrict the gauge transformations $e^{-i\Omega}$ to compactly supported Ω as well. Treating more general vector potentials A and gauge transforms $e^{-i\Omega}$ would require an analysis of decay properties at infinity which is not our focus here.

Finally, given Cauchy surface Σ , there is an explicit representative of the equivalence class of polarizations $\mathcal{C}_\Sigma(A)$ which can be given in terms of a compact, skew-adjoint linear operator $Q_\Sigma^A : \mathcal{H}_\Sigma \hookrightarrow \mathcal{H}_\Sigma$, as defined in (56) below. With it the polarization class can be identified as follows:

Theorem 1.7. *Given Cauchy surface Σ , we have $\mathcal{C}_\Sigma(A) = \left[e^{Q_\Sigma^A} \mathcal{H}_\Sigma^- \right]_\approx$.*

Other representatives for polarization classes $\mathcal{C}_\Sigma(A)$ beyond the “interpolating representation” $U_{\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^-$, as used in Definition 1.3, can be inferred from the so-called Furry picture, as worked out for equal-time hyperplanes in [6], and from the global constructions of the fermionic projector given in [11, 10]. In contrast to global constructions, the representation given in Theorem 1.7 uses only *local* geometric information of the vector potential A at Σ ; cf. (56), (39), and Lemma 2.3 below.

The implications on the physical picture can be seen as follows. The Dirac sea on Cauchy surface Σ can be described in any Fock space $\mathcal{F}(V, \mathcal{H}_\Sigma)$ for any choice of polarization $V \in \mathcal{C}_\Sigma(A)$. The polarization class $\mathcal{C}_\Sigma(A)$ is uniquely determined by the tangential components of the external potential A on Σ . This is an object that transforms covariantly under Lorentz and gauge transformations. The choice of the particular polarization can then be seen as a “choice of coordinates” in which the Dirac sea is described. When regarding the Dirac evolution from one Cauchy surface Σ to Σ' another “choice of coordinates” $W \in \mathcal{C}_{\Sigma'}(A)$ has to be made. Then one yields an evolution operator $\tilde{U}_{\Sigma\Sigma'}^A : \mathcal{F}(V, \mathcal{H}_\Sigma) \rightarrow \mathcal{F}(W, \mathcal{H}_{\Sigma'})$ which is unique up to an arbitrary phase Corollary 1.4. Transition probabilities of the kind $|\langle \Psi, \tilde{U}_{\Sigma\Sigma'}^A \Phi \rangle|^2$ for $\Psi \in \mathcal{F}(W, \mathcal{H}_{\Sigma'})$ and $\Phi \in \mathcal{F}(V, \mathcal{H}_\Sigma)$ are well-defined and unique without

the need of a renormalization method. Finally, for a family of Cauchy surfaces $(\Sigma_t)_{t \in \mathbb{R}}$ that interpolates smoothly between Σ and Σ' we also give an infinitesimal version of how the external potential A changes the polarization in terms of the flow parameter t ; see Theorem 2.8 below.

Remark 1.8 (Charge Sectors). *Given two polarizations $V, W \in \text{Pol}(\mathcal{H}_\Sigma)$ such that $P_\Sigma^V - P_\Sigma^W$ is a compact operator, e.g., as in the case $V \approx W$ as defined in (8), one can define their relative charge, denoted by $\text{charge}(V, W)$, to be the Fredholm index of $P_\Sigma^W|_{V \rightarrow W}$; cf. [2]. The equivalence relation \approx in the claim of Theorem 1.7 can then be replaced by the finer equivalence relation \approx_0 , which is defined as follows: $V \approx_0 W$ if and only if $V \approx W$ and $\text{charge}(V, W) = 0$. This is shown as an addendum to the proof of Theorem 1.7.*

1.2 Outlook

As indicated at the end of the introduction the current operator depends directly on the unspecified phase of $\tilde{U}_{\Sigma'\Sigma}^A$. This can be seen from Bogolyubov's formula

$$j^\mu(x) = i\tilde{U}_{\Sigma_{\text{in}}\Sigma_{\text{out}}}^A \frac{\delta \tilde{U}_{\Sigma_{\text{out}}\Sigma_{\text{in}}}^A}{\delta A_\mu(x)} \quad (12)$$

where Σ_{out} is a Cauchy surfaces in the remote future of the support of A such that $\Sigma_{\text{out}} \cap \text{supp } A = \emptyset$. Hence, without identification of the derivative of the phase of $\tilde{U}_{\Sigma'\Sigma}^A$ the physical current is not fully specified. Nevertheless, now the situation is slightly better than in the standard perturbative approach. As for each choice of admissible polarizations in $\mathcal{C}_{\Sigma'}(A)$ and $\mathcal{C}_\Sigma(A)$, identified above, there is a well-defined lift $\tilde{U}_{\Sigma'\Sigma}^A$ of the Dirac evolution operator $U_{\Sigma'\Sigma}^A$ and therefore also a well-defined current (12). Now it is only the task to select the physical relevant one. One way of doing so is to impose extra conditions on the (12), and hence, the phase, so that the set of admissible phases shrinks to one that produces the same currents up to the known freedom of charge renormalization; see [5, 19, 15, 12]. In the case of equal-time hyperplanes a choice of the unidentified phase was given by parallel transport in [16]. On top of the geometric construction and despite the fact that there are still degrees of freedom left, Mickelsson's current is particularly interesting because it agrees with conventional perturbation theory up to second order. Yet the open question remains which additional physical requirements may constraint these degree of freedoms up to the one of the numerical value of the elementary charge e fixed by the experiment.

The issue of the unidentified phase particularly concerns the so-called phenomenon of “vacuum polarization” as well as the dynamical description of pair creation processes for which only a few rigorous treatments are available; e.g., see [13] for vacuum polarization in the Hartree-Fock approximation for static external sources, [17] for adiabatic pair creation, and for a more fundamental approach the so-called “Theory of Causal Fermion Systems” [7, 8, 9], which is based on a reformulation of quantum electrodynamics by means of an action principle.

1.3 Definitions, Constants, Notation, and previous Results

In this section we briefly review the notation and results about the one-particle Dirac evolution on Cauchy surfaces provided in a previous work [3]. The present article, dealing with

the second-quantization Dirac evolution, is based on this work.

Space-time \mathbb{R}^4 is endowed with metric tensor $g = (g_{\mu\nu})_{\mu,\nu=0,1,2,3} = \text{diag}(1, -1, -1, -1)$, and its elements are denoted by four-vectors $x = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}) = x^\mu e_\mu$, for e_μ being the canonical basis vectors. Raising and lowering of indices is done w.r.t. g . Moreover, we use Einstein's summation convention, the standard representation of the Dirac matrices $\gamma^\mu \in \mathbb{C}^{4 \times 4}$ that fulfill $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and Feynman's slash-notation $\not{\partial} = \gamma^\mu \partial_\mu$, $\not{A} = \gamma^\mu A_\mu$. When considering subsets of space-time \mathbb{R}^4 we shall use the following notations: Causal $:= \{x \in \mathbb{R}^4 \mid x_\mu x^\mu \geq 0\}$ and Past $:= \{x \in \mathbb{R}^4 \mid x_\mu x^\mu > 0, x^0 < 0\}$.

The central geometric objects for posing the initial value problem for (1) are Cauchy surfaces defined as follows:

Definition 1.9 (Cauchy Surfaces). *We define a Cauchy surface Σ in \mathbb{R}^4 to be a smooth, 3-dimensional submanifold of \mathbb{R}^4 that fulfills the following three conditions:*

- (a) *Every inextensible, two-sided, time- or light-like, continuous path in \mathbb{R}^4 intersects Σ in a unique point.*
- (b) *For every $x \in \Sigma$, the tangential space $T_x \Sigma$ is space-like.*
- (c) *The tangential spaces to Σ are bounded away from light-like directions in the following sense: The only light-like accumulation point of $\bigcup_{x \in \Sigma} T_x \Sigma$ is zero.*

In coordinates, every Cauchy surface Σ can be parametrized as

$$\Sigma = \{\pi_\Sigma(\mathbf{x}) := (t_\Sigma(\mathbf{x}), \mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^3\} \quad (13)$$

with a smooth function $t_\Sigma : \mathbb{R}^3 \rightarrow \mathbb{R}$. For convenience and without restricting generality of our results we keep a global constant

$$0 < V_{\max} < 1 \quad (14)$$

fixed and work only with Cauchy surfaces Σ such that

$$\sup_{\mathbf{x} \in \mathbb{R}^3} |\nabla t_\Sigma(\mathbf{x})| \leq V_{\max}. \quad (15)$$

The assumption (c) in Definition 1.9 and (15) can be relaxed to $|\nabla t_\Sigma(\mathbf{x})| < 1$ for all $\mathbf{x} \in \mathbb{R}^3$ due to the causal structure of the solutions to the Dirac equation, although this is not worked out in this paper.

The standard volume form over \mathbb{R}^4 is denoted by $d^4x = dx^0 dx^1 dx^2 dx^3$; the product of forms is understood as wedge product. The symbols d^3x and $d^3\mathbf{x}$ mean the 3-form $d^3x = dx^1 dx^2 dx^3$ on \mathbb{R}^4 and on \mathbb{R}^3 , respectively. Contraction of a form ω with a vector v is denoted by $i_v(\omega)$. The notation $i_v(\omega)$ is also used for the spinor matrix valued vector $\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = \gamma^\mu e_\mu$:

$$i_\gamma(d^4x) = \gamma^\mu i_{e_\mu}(d^4x). \quad (16)$$

Furthermore, for a 4-spinor $\psi \in \mathbb{C}^4$ (viewed as column vector), $\bar{\psi}$ stands for the row vector $\psi^* \gamma^0$, where $*$ denotes hermitian conjugation.

Smooth families $(\Sigma_t)_{t \in T}$ of Cauchy surfaces, indexed by an interval $T \subseteq \mathbb{R}$ and fulfilling (15), are denoted by

$$\Sigma := \{(x, t) \mid t \in T, x \in \Sigma_t\}. \quad (17)$$

Given the external electromagnetic vector potential $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ of interest, we assume that the set $\{(x, t) \in \Sigma \mid x \in \text{supp}(A)\}$ is compact. This condition is trivially fulfilled in the important case of a compact interval $T = [t_0, t_1]$ with Σ interpolating between two Cauchy surfaces Σ_{t_0} and Σ_{t_1} . The compactness condition is also automatically fulfilled in the case that $T = \mathbb{R}$ with Σ being a smooth foliation of the Minkowski space-time \mathbb{R}^4 .

We assume furthermore that the family $(\Sigma_t)_{t \in T}$ is driven by a (Minkowski) normal vector field $vn : \Sigma \rightarrow \mathbb{R}^4$, where $n : \Sigma \rightarrow \mathbb{R}^4$, $(x, t) \mapsto n_t(x)$, denotes the future-directed (Minkowski) normal unit vector field to the Cauchy surfaces and $v : \Sigma \rightarrow \mathbb{R}$, $(x, t) \mapsto v_t(x)$, denotes the speed at which the Cauchy surfaces move forward in normal direction. For technical reasons, in particular when using the chain rule, it is convenient to extend the “speed” v and the unit vector field n in a smooth way to the domain $\mathbb{R}^4 \times T$. In the case that Σ is a foliation of space-time, we may even drop the t -dependence of v and n . In this important case, some of the arguments below become slightly simpler.

Definition 1.10 (Spaces of Initial Data). *For any Cauchy surface Σ we define the vector space $\mathcal{C}_\Sigma := C_c^\infty(\Sigma, \mathbb{C}^4)$. For a given Cauchy surface Σ , let $\mathcal{H}_\Sigma = L^2(\Sigma, \mathbb{C}^4)$ denote the vector space of all 4-spinor valued measurable functions $\phi : \Sigma \rightarrow \mathbb{C}^4$ (modulo changes on null sets) having a finite norm $\|\phi\| = \sqrt{\langle \phi, \phi \rangle} < \infty$ w.r.t. the scalar product*

$$\langle \phi, \psi \rangle = \int_\Sigma \overline{\phi(x)} i_\gamma(d^4x) \psi(x). \quad (18)$$

For $x \in \Sigma$, the restriction of the spinor matrix valued 3-form $i_\gamma(d^4x)$ to the tangential space $T_x \Sigma$ is given by

$$i_\gamma(d^4x) = \not{n}(x) i_n(d^4x) = \left(\gamma^0 - \sum_{\mu=1}^3 \gamma^\mu \frac{\partial t_\Sigma(\mathbf{x})}{\partial x^\mu} \right) d^3x =: \Gamma(\mathbf{x}) d^3x \text{ on } (T_x \Sigma)^3. \quad (19)$$

As a consequence of the (15), there is a positive constant $\Gamma_{\max} = \Gamma_{\max}(V_{\max})$ such that

$$\|\Gamma(\mathbf{x})\| \leq \Gamma_{\max}, \quad \forall \mathbf{x} \in \mathbb{R}^3. \quad (20)$$

The class of solutions to the Dirac equation (1) considered in this work is defined by:

Definition 1.11 (Solution Spaces).

- (i) Let \mathcal{C}_A denote the space of all smooth solutions $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ of the Dirac equation (1) which have a spatially compact causal support in the following sense: There is a compact set $K \subset \mathbb{R}^4$ such that $\text{supp } \psi \subseteq K + \text{Causal}$.
- (ii) We endow \mathcal{C}_A with the scalar product given in (18); note that due to conservation of the 4-vector current $\bar{\phi} \gamma^\mu \psi$, the scalar product $\langle \cdot, \cdot \rangle : \mathcal{C}_A \times \mathcal{C}_A \rightarrow \mathbb{C}$ is independent of the particular choice of Σ .

(iii) Let \mathcal{H}_A be the Hilbert space given by the (abstract) completion of \mathcal{C}_A .

Theorem 2.21 in [3] ensures:

Theorem 1.12 (Initial Value Problem and Support). *Let Σ be a Cauchy surface and $\chi_\Sigma \in \mathcal{C}_c^\infty(\Sigma, \mathbb{C}^4)$ be given initial data. Then, there is a $\psi \in \mathcal{C}_A$ such that $\psi|_\Sigma = \chi_\Sigma$ and $\text{supp } \psi \subseteq \text{supp } \chi_\Sigma + \text{Causal}$. Moreover, suppose $\tilde{\psi} \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ solves the Dirac equation (1) for initial data $\tilde{\psi}|_\Sigma = \chi_\Sigma$, then $\tilde{\psi} = \psi$.*

This theorem gives rise to the following definition in which we use the notation $\psi|_\Sigma \in \mathcal{C}_\Sigma$ to denote the restriction of a $\psi \in \mathcal{C}_A$ to a Cauchy surface Σ .

Definition 1.13 (Evolution Operators). *Let Σ, Σ' be Cauchy surfaces. In view of Theorem 1.12 we define the isomorphic isometries*

$$\begin{aligned} U_{\Sigma A} : \mathcal{C}_A &\rightarrow \mathcal{C}_\Sigma, & U_{\Sigma A} \phi &:= \phi|_\Sigma, \\ U_{A\Sigma} : \mathcal{C}_\Sigma &\rightarrow \mathcal{C}_A, & U_{\Sigma A} \chi_\Sigma &:= \psi, \\ U_{\Sigma'\Sigma}^A : \mathcal{C}_\Sigma &\rightarrow \mathcal{C}_{\Sigma'}, & U_{\Sigma'\Sigma}^A &:= U_{\Sigma' A} U_{A\Sigma}, \end{aligned} \tag{21}$$

where $\chi_\Sigma \in \mathcal{C}_\Sigma$, $\phi \in \mathcal{C}_A$, and ψ is the solution corresponding to initial value χ_Σ as in Theorem 1.12. These maps extend uniquely to unitary maps $U_{A\Sigma} : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_A$, $U_{\Sigma A} : \mathcal{H}_A \rightarrow \mathcal{H}_\Sigma$ and $U_{\Sigma'\Sigma}^A : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\Sigma'}$.

Here we differ from the notation used in Theorem 2.23 in [3] where $U_{\Sigma'\Sigma}^A$ was denoted by $\mathcal{F}_{\Sigma'\Sigma}^A$. Furthermore, it will be useful to express the orthogonal projector P_Σ^- in an momentum integral representation over the mass shell

$$\mathcal{M} = \{p \in \mathbb{R}^4 \mid p_\mu p^\mu = m^2\} = \mathcal{M}_+ \cup \mathcal{M}_-, \quad \mathcal{M}_\pm = \{p \in \mathcal{M} \mid \pm p^0 > 0\}; \tag{22}$$

cf. Lemma 2.1 and the definition of $\mathcal{F}_{\mathcal{M}\Sigma}$ in [3]. We endow \mathcal{M} with the orientation that makes the projection $\mathcal{M} \rightarrow \mathbb{R}^3$, $(p^0, \mathbf{p}) \mapsto \mathbf{p}$ positively oriented. One finds that

$$i_p(d^4 p) = \frac{m^2}{p^0} dp^1 dp^2 dp^3 = \frac{m^2}{p^0} d^3 p \text{ on } (T_p \mathcal{M})^3. \tag{23}$$

General Notation. Positive constants and remainder terms are denoted by C_1, C_2, C_3, \dots and r_1, r_2, r_3, \dots , respectively. They keep their meaning throughout the whole article. Any fixed quantity a constant depends on (except numerical constants like electron mass m and charge e) is displayed at least once when the constant is introduced. Furthermore, we classify the behavior of functions using the following variant of the Landau symbol notation.

Definition 1.14. *For lists of variables x, y, z we use the notation*

$$f(x, y, z) = O_y(g(x)), \quad \text{for all } (x, y, z) \in \text{domain} \tag{24}$$

to mean the following: There exists a constant $C(y)$ depending only on the parameters y , but neither on x nor on z , such that

$$|f(x, y, z)| \leq C(y)|g(x)|, \quad \text{for all } (x, y, z) \in \text{domain}, \tag{25}$$

where $|\cdot|$ stands for the appropriate norm applicable to f . Note that the notation (24) does not mean that $f(x, y, z) = f(x, y)$, i.e., that the value of f is independent of z . Rather, it just means that the bound is uniform in z .

2 Proofs

The key idea in the proofs of our main results Theorem 1.5, 1.6, and 1.7 is to guess a simple enough operator $P_\Sigma^A : \mathcal{H}_\Sigma \hookrightarrow$ so that

$$U_{\Sigma\Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^- U_{\Sigma_{\text{in}}\Sigma}^A - P_\Sigma^A \in I_2(\mathcal{H}_\Sigma). \quad (26)$$

It turns out that all claims about the properties of the polarization classes $\mathcal{C}_\Sigma(A)$ above can then be inferred from the properties of P_Σ^A . This is due to the fact that (26) is compatible with the Hilbert-Schmidt operator freedom encoded in the \approx equivalence relation that was used to define the polarization classes $\mathcal{C}_\Sigma(A)$; see Definition 1.3.

The intuition behind our guess of P_Σ^A comes from gauge transforms. Imagine the special situation in which an external potential A could be gauged to zero, i.e., $A = \partial\Omega$ for a given scalar field Ω . In this case $e^{-i\Omega} P_\Sigma^- e^{i\Omega}$ is a good candidate for P_Σ^A . Now in the case of general external potentials A that cannot be attained by a gauge transformation of the zero potential, the idea is to implement different gauge transforms locally near to each space-time point. For example, if $p^-(y-x)$ denotes the informal integral kernel of the operator P_Σ^- , one could try to define P_Σ^A as the operator corresponding to the informal kernel $p^A(x, y) = e^{-i\lambda^A(x, y)} p^-(y-x)$ for the choice $\lambda^A(x) = \frac{1}{2}(A(x) + A(y))_\mu(y-x)^\mu$. Due to this choice, the action of $\lambda^A(x, y)$ can be interpreted as a local gauge transform of $p^-(y-x)$ from the zero potential to the potential $A_\mu(x)$ at space-time point x . It turns out that these local gauge transforms give rise to an operator P_Σ^A that fulfills (26).

Section Overview In Section 2.1 we define the operators P_Σ^- and P_Σ^A and state their main properties. Assuming these properties we prove our main results in Section 2.2. The proofs of those employed properties are delivered afterwards in Sections 2.3 and 2.4.

2.1 The Operators P_Σ^- and P_Σ^A

As described in the previous section, the central objects of our study are the operators P_Σ^- and operators which are derived from them like the discussed P_Σ^A . Lemma 2.1 describes the integral representation of the orthogonal projector P_Σ^- . For this we introduce the notation

$$r(w) := \sqrt{-w_\mu w^\mu} \quad \text{for} \quad w \in \text{domain}(r) := \{w \in \mathbb{C}^4 \mid -w_\mu w^\mu \in \mathbb{C} \setminus \mathbb{R}_0^-\}. \quad (27)$$

The square root is interpreted as its principal value $\sqrt{r^2 e^{2i\varphi}} = r e^{i\varphi}$ for $r > 0$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. We note that for a Cauchy surface Σ fulfilling (15) and $0 \neq z = y - x$ with $x, y \in \Sigma$ one has

$$\sqrt{1 - V_{\text{max}}^2} |z| \leq r(z) \leq |z| \leq |z| \leq \sqrt{1 + V_{\text{max}}^2} |z|. \quad (28)$$

To deal with the singularity of the informal integral kernel $p^-(y-x)$ of the projection operator P_Σ^- at the diagonal $x = y$, we use a regularization shifting the argument $y - x$ a little in direction of the imaginary past.

Lemma 2.1. *For $\phi, \psi \in \mathcal{C}_\Sigma$ and any past-directed time-like vector $u \in \text{Past}$ one has*

$$\langle \phi, P_\Sigma^- \psi \rangle = \lim_{\epsilon \downarrow 0} \int_{x \in \Sigma} \bar{\phi}(x) i_\gamma(d^4 x) \int_{y \in \Sigma} p^-(y - x + i\epsilon u) i_\gamma(d^4 y) \psi(y), \quad (29)$$

where

$$p^- : \mathbb{R}^4 + i \text{Past} \rightarrow \mathbb{C}^{4 \times 4}, \quad p^-(w) = \frac{1}{(2\pi)^3 m} \int_{\mathcal{M}_-} \frac{\not{p} + m}{2m} e^{ipw} i_p(d^4 p) = \frac{-i\not{p} + m}{2m} D(w), \quad (30)$$

$$D : \mathbb{R}^4 + i \text{Past} \rightarrow \mathbb{C}, \quad D(w) = \frac{1}{(2\pi)^3 m} \int_{\mathcal{M}_-} e^{ipw} i_p(d^4 p) = -\frac{m^3}{2\pi^2} \frac{K_1(mr(w))}{mr(w)}, \quad (31)$$

$$K_1 : \mathbb{R}^+ + i\mathbb{R} \rightarrow \mathbb{C}, \quad K_1(\xi) = \xi \int_1^\infty e^{-\xi s} \sqrt{s^2 - 1} ds. \quad (32)$$

K_1 is the modified Bessel function of second kind of order one. The functions D and p^- have analytic continuations defined on $\text{domain}(r)$. The corresponding continuations are denoted by the same symbols.

The proof is given in Section 2.3. It is based on the momentum integral representation given in Theorem 2.15 in [3]. In the following we define several candidates for P_Σ^A fulfilling the key property (26) as discussed in the beginning of Section 2. We will denote these operators by $P_\Sigma^\lambda : \mathcal{H}_\Sigma \hookrightarrow$ where the superscript λ denotes an element out of the following class of “local” gauge functions:

Definition 2.2. For $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ let $\mathcal{G}(A)$ denote the set of all functions $\lambda : \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}$ with the following properties:

- (i) $\lambda \in C^\infty(\mathbb{R}^4 \times \mathbb{R}^4, \mathbb{R})$.
- (ii) There is a compact set $K \subset \mathbb{R}^4$ such that $\text{supp } \lambda \subseteq K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$.
- (iii) λ vanishes on the diagonal, i.e., $\lambda(x, x) = 0$ for $x \in \mathbb{R}^4$.
- (iv) On the diagonal the first derivatives fulfill

$$\partial^x \lambda(x, y) = -\partial^y \lambda(x, y) = A(x) \quad \text{for } x = y \in \mathbb{R}^4. \quad (33)$$

Given a “local” gauge transform $\lambda \in \mathcal{G}(A)$ we define the corresponding operator P_Σ^λ using the heuristic idea behind P_Σ^A we discussed in the beginning of Section 2.

Lemma 2.3. Given $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and $\lambda \in \mathcal{G}(A)$ there is a unique bounded operator $P_\Sigma^\lambda : \mathcal{H}_\Sigma \hookrightarrow$ with matrix elements

$$\langle \phi, P_\Sigma^\lambda \psi \rangle = \lim_{\epsilon \downarrow 0} \langle \phi, P_\Sigma^{\lambda, \epsilon u} \psi \rangle \quad \text{with} \quad (34)$$

$$\langle \phi, P_\Sigma^{\lambda, \epsilon u} \psi \rangle := \int_{x \in \Sigma} \bar{\phi}(x) i_\gamma(d^4 x) \int_{y \in \Sigma} e^{-i\lambda(x, y)} p^-(y - x + i\epsilon u) i_\gamma(d^4 y) \psi(y). \quad (35)$$

for any given $\phi, \psi \in \mathcal{C}_\Sigma$ and any past-directed time-like vector $u \in \text{Past}$. In particular, the limit in (34) does not depend on the choice of $u \in \text{Past}$. For $\Delta P_\Sigma^\lambda := P_\Sigma^\lambda - P_\Sigma^-$, $\psi \in \mathcal{H}_\Sigma$, and almost all $x \in \Sigma$ it holds

$$(\Delta P_\Sigma^\lambda \psi)(x) = \int_{y \in \Sigma} (e^{-i\lambda(x, y)} - 1) p^-(y - x) i_\gamma(d^4 y) \psi(y), \quad (36)$$

and furthermore:

- (i) The operator norm of P_Σ^λ is bounded by a constant $C_1(V_{\max}, \lambda)$; cf. (15);
- (ii) ΔP_Σ^λ is a compact operator;
- (iii) $|\Delta P_\Sigma^\lambda|^2$ is a Hilbert-Schmidt operator.
- (iv) If $\lambda(x, y) = -\lambda(y, x)$ for all $x, y \in \Sigma$, then P_Σ^λ is self-adjoint.

This lemma is proven in Section 2.3. Two important examples of elements in $\mathcal{G}(A)$ are:

- The choice $\lambda(x, y) = \Omega(x) - \Omega(y)$ for $\Omega \in C_c^\infty(\mathbb{R}^4, \mathbb{R})$ fulfills $\lambda \in \mathcal{G}(\partial\Omega)$. Such a λ delivers a good candidate for the operator P_Σ^A fulfilling (26) if the external field A can be attained from the zero field via a gauge transform $A = 0 \mapsto A = \partial\Omega$. We observe for any path $C_{y,x}$ from y to x

$$\lambda(x, y) = \int_{C_{y,x}} A_\mu(u) du^\mu = \frac{1}{2}(A_\mu(x) + A_\mu(y))(x^\mu - y^\mu) + O_A(|x - y|^3). \quad (37)$$

- For an arbitrary vector potential $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ also

$$\lambda^A(x, y) := \frac{1}{2}(A_\mu(x) + A_\mu(y))(x^\mu - y^\mu) \quad (38)$$

fulfills $\lambda^A \in \mathcal{G}(A)$. This choice is motivated by the special case (37). It will be particularly convenient for our work. Note that it has the symmetry $\lambda^A(x, y) = -\lambda^A(y, x)$; cf. part (iv) in Lemma 2.3. In particular, the operator P_Σ^A from the discussion will be given by

$$P_\Sigma^A := P_\Sigma^{\lambda^A}. \quad (39)$$

We shall show that for $\lambda \in \mathcal{G}(A)$ the operators P_Σ^λ obey the key property (26). Our first result about P_Σ^λ for a $\lambda \in \mathcal{G}(A)$ is that, up to a Hilbert-Schmidt operator, it depends only on the restriction of the 1-form A to the tangent bundle $T\Sigma$ of the Cauchy surface Σ .

Theorem 2.4. *Given $A, \tilde{A} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and $\lambda \in \mathcal{G}(A)$, $\tilde{\lambda} \in \mathcal{G}(\tilde{A})$, the following is true:*

$$P_\Sigma^\lambda - P_\Sigma^{\tilde{\lambda}} \in I_2(\mathcal{H}_\Sigma) \quad \Leftrightarrow \quad A|_{T\Sigma} = \tilde{A}|_{T\Sigma}. \quad (40)$$

This theorem is also proven in Section 2.3. From our next result we can infer that the operators P_Σ^λ obey the key property (26).

Theorem 2.5. *Given $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$, $\lambda \in \mathcal{G}(A)$, and two Cauchy surfaces Σ, Σ' , one has*

$$U_{A\Sigma'} P_{\Sigma'}^\lambda U_{\Sigma'A} - U_{A\Sigma} P_\Sigma^\lambda U_{\Sigma A} \in I_2(\mathcal{H}_A), \quad (41)$$

where $U_{A\Sigma}$ and $U_{\Sigma A}$ are the Dirac evolution operators given in Definition 1.13.

Instead of proving this theorem directly we prove it at the end of Section 2.4 as consequence of Theorem 2.8 below. The latter can be understood as an infinitesimal version of Theorem 2.5. To state Theorem 2.8 we consider a family $(\Sigma_t)_{t \in T}$ of Cauchy surfaces encoded by Σ , see (17), such that $\Sigma = \Sigma_{t_0}$ and $\Sigma' = \Sigma_{t_1}$. In addition we need the following helper object s_Σ^A defined in Definition 2.6 below as well as the following notation. Given an electromagnetic potential $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$ and a Cauchy surface Σ with future-directed unit normal vector field n , we define the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$E_\mu := F_{\mu\nu} n^\nu \quad (42)$$

referred to as the “electric field” with respect to the local Cauchy surface Σ . In the special case $n = e_0 = (1, 0, 0, 0)$, this encodes just the electric part of the electromagnetic field tensor.

Recall from the paragraph preceding Definition 1.10 that we extended the unit normal field n on the Cauchy surface to a smooth unit normal field $n : \mathbb{R}^4 \times T \rightarrow \mathbb{R}^4$ and velocity field $v : \mathbb{R}^4 \times T \rightarrow \mathbb{R}$, which induces the “electric field” E to be defined on $\mathbb{R}^4 \times T$ as well. In particular, after this extension, the partial derivative $\partial E_\mu(x, t)/\partial t = F_{\mu\nu}(x) \partial n_t^\nu(x)/\partial t$ then makes sense.

Definition 2.6. Recall the definitions of $r(w)$ and $D(w)$ given in (27) and (31), respectively. For $\epsilon > 0$, $u \in \text{Past}$, and $x, y \in \mathbb{R}^4$, we define the integral kernel

$$s_\Sigma^{A, \epsilon u}(x, y) := \frac{1}{8m} \not{n}(x) \not{E}(x) r(w)^2 \not{\partial} D(w), \quad \text{where } w = y - x + i\epsilon u. \quad (43)$$

Furthermore, for $x - y$ being space-like (in particular $x \neq y$), we also define the integral kernel

$$s_\Sigma^A(x, y) = s_\Sigma^{A, 0}(x, y) := \lim_{\epsilon \downarrow 0} s_\Sigma^{A, \epsilon u}(x, y) = \frac{1}{8m} \not{n}(x) \not{E}(x) r(y - x)^2 \not{\partial} D(y - x). \quad (44)$$

We remark that restricted to x and y within a single Cauchy surface Σ , the value of the kernel $s_\Sigma^{A, \epsilon u}(x, y)$ depends only on Σ through its normal field $n : \Sigma \rightarrow \mathbb{R}^4$. In this case the definition makes sense without specifying neither the velocity field v nor the extension of n and v to $\mathbb{R}^4 \times T$. In particular, $s_\Sigma^{A, \epsilon u}(x, y)$ depends only on the Cauchy surface Σ but not on the choice of a family $(\Sigma_t)_{t \in T}$. This stands in contrast to the derivative $\partial s_\Sigma^{A, \epsilon u}/\partial t$, which makes sense everywhere only given a family $(\Sigma_t)_{t \in T}$ and the extended version of n .

Exploiting the properties of $D(w)$ given in Lemma 2.1 and in Corollary A.1 in the appendix we shall find:

Lemma 2.7. Let $u \in \text{Past}$.

(i) The integral kernels $s_\Sigma^{A, \epsilon u}$, $\epsilon \geq 0$, give rise to Hilbert-Schmidt operators

$$S_\Sigma^{A, \epsilon u} : \mathcal{H}_\Sigma \hookrightarrow, \quad S_\Sigma^{A, \epsilon u} \psi(x) := \int_\Sigma s_\Sigma^{A, \epsilon u}(x, y) i_\gamma(d^4 y) \psi(y) \quad \text{for almost all } x \in \Sigma, \quad (45)$$

$$S_\Sigma^A := S_\Sigma^{A, 0}, \text{ with the property that } \|S_\Sigma^A - S_\Sigma^{A, \epsilon u}\|_{I_2(\mathcal{H}_\Sigma)} \xrightarrow{\epsilon \downarrow 0} 0.$$

(ii) Similarly, for $t \in T$, the integral kernels $\partial s_{\Sigma_t}^{A, \epsilon u} / \partial t$, $\epsilon \geq 0$, give rise to Hilbert-Schmidt operators

$$\dot{S}_{\Sigma_t}^{A, \epsilon u} : \mathcal{H}_{\Sigma} \hookrightarrow, \quad \dot{S}_{\Sigma_t}^{A, \epsilon u} \psi(x) := \int_{\Sigma_t} \frac{\partial s_{\Sigma_t}^{A, \epsilon u}}{\partial t}(x, y) i_{\gamma}(d^4 y) \psi(y) \quad \text{for almost all } x \in \Sigma_t, \quad (46)$$

$\dot{S}_{\Sigma_t}^A := \dot{S}_{\Sigma_t}^{A, 0}$, with the property that $\sup_{t \in T} \|\dot{S}_{\Sigma_t}^A\|_{I_2(\mathcal{H}_{\Sigma_t})} < \infty$ and $\|\dot{S}_{\Sigma_t}^A - \dot{S}_{\Sigma_t}^{A, \epsilon u}\|_{I_2(\mathcal{H}_{\Sigma_t})} \xrightarrow{\epsilon \downarrow 0} 0$ for all t .

With this ingredient our infinitesimal version of Theorem 2.5 can be formulated as follows; for technical convenience, we phrase it only for the special choice $\lambda^A \in \mathcal{G}(A)$ defined in (38).

Theorem 2.8. *Given $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$, any smooth family of Cauchy surfaces Σ , cf. (17), and $t_0, t_1 \in T$, and one has*

$$U_{A\Sigma_{t_1}} \left(P_{\Sigma_{t_1}}^A + S_{\Sigma_{t_1}}^A \right) U_{\Sigma_{t_1}A} - U_{A\Sigma_{t_0}} \left(P_{\Sigma_{t_0}}^A + S_{\Sigma_{t_0}}^A \right) U_{\Sigma_{t_0}A} = \int_{t_0}^{t_1} U_{A\Sigma_t} R(t) U_{\Sigma_t A} dt \quad (47)$$

for a family of Hilbert-Schmidt operators $R(t)$, $t \in T$, with $\sup_{t \in T} \|R(t)\|_{I_2(\mathcal{H}_{\Sigma_t})} < \infty$. The integral in (47) is understood in the weak sense.

Note that for the choice $\lambda \in \mathcal{G}(A)$, $\Sigma_{t_1} = \Sigma$, $\Sigma_{t_0} = \Sigma_{\text{in}}$ one has $P_{\Sigma_{\text{in}}}^\lambda = P_{\Sigma_{\text{in}}}^-$, and the restriction of (41) to Cauchy surface Σ yields property $U_{\Sigma\Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^- U_{\Sigma_{\text{in}}\Sigma}^A - P_{\Sigma}^\lambda \in I_2(\mathcal{H}_{\Sigma})$, i.e., the key property (26). The proof of Theorem 2.8 given in Section 2.4 is the heart of this work.

2.2 Proofs of Main Results

In this section, we prove the main results under the assumption that the claims in Section 2.1 are true. The proofs of these assumed claims are then provided in Sections 2.3-2.4. The connection of how to infer the properties of $\mathcal{C}_{\Sigma}(A)$ from the properties of the operators P_{Σ}^λ is given by the following lemma.

Lemma 2.9. *Let $A \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$, Σ be a Cauchy surface, and $\lambda \in \mathcal{G}(A)$. Then for every polarization V in \mathcal{H}_{Σ} , we have*

$$V \in \mathcal{C}_{\Sigma}(A) \quad \Leftrightarrow \quad P_{\Sigma}^V - P_{\Sigma}^\lambda \in I_2(\mathcal{H}_{\Sigma}). \quad (48)$$

Proof. By Definition 1.3, $V \in \mathcal{C}_{\Sigma}(A)$ is equivalent to

$$P_{\Sigma}^V - U_{\Sigma\Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^- U_{\Sigma_{\text{in}}\Sigma}^A \in I_2(\mathcal{H}_{\Sigma}). \quad (49)$$

On the other hand, Theorem 2.5 implies

$$P_{\Sigma}^\lambda - U_{\Sigma\Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^- U_{\Sigma_{\text{in}}\Sigma}^A \in I_2(\mathcal{H}_{\Sigma}). \quad (50)$$

Thus, statement (49) is equivalent to $P_{\Sigma}^V - P_{\Sigma}^\lambda \in I_2(\mathcal{H}_{\Sigma})$. \square

Proof of Theorem 1.5. $\mathcal{C}_\Sigma(A) = \mathcal{C}_\Sigma(\tilde{A})$ holds true if and only if there are $V \in \mathcal{C}_\Sigma(A)$ and $W \in \mathcal{C}_\Sigma(\tilde{A})$ such that

$$P_\Sigma^V - P_\Sigma^W \in I_2(\mathcal{H}_\Sigma). \quad (51)$$

Let $\lambda \in \mathcal{G}(A)$ and $\tilde{\lambda} \in \mathcal{G}(\tilde{A})$. In view of Lemma 2.9, statement (51) is equivalent to $P_\Sigma^\lambda - P_\Sigma^{\tilde{\lambda}} \in I_2(\mathcal{H}_\Sigma)$. Due to Theorem 2.4 the latter is equivalent to $A|_{T\Sigma} = \tilde{A}|_{T\Sigma}$, which proves the claim. \square

Proof of Theorem 1.6. Claim (i): It is sufficient to prove that there exist $V \in \mathcal{C}_\Sigma(A)$ and $W \in \mathcal{C}_{\Lambda\Sigma}(\Lambda A(\Lambda^{-1}\cdot))$ such that $L^{(S,\Lambda)} P_\Sigma^V (L^{(S,\Lambda)})^{-1} - P_{\Lambda\Sigma}^W \in I_2(\mathcal{H}_{\Lambda\Sigma})$. We remark that for the linear form A , ΛA stands for the linear form with coordinates $\Lambda_\mu^\nu A_\nu$, while for a vector x , the term Λx stands for the vector with coordinates $\Lambda_\mu^\nu x^\nu$. We take $\lambda \in \mathcal{G}(A)$, e.g., $\lambda = \lambda^A$ from (38). Thanks to Lemma 2.9, for all $V \in \mathcal{C}_\Sigma(A)$ we have $P_\Sigma^V - P_\Sigma^\lambda \in I_2(\mathcal{H}_\Sigma)$. First, let us discuss how such a P_Σ^λ behaves under the Lorentz transforms $L^{(S,\Lambda)}$. For $\epsilon > 0$ and $u \in \text{Past}$, the integral kernel $p_\Sigma^{\lambda, \epsilon u}(x, y) = e^{-i\lambda(x, y)} p_-(y - x + i\epsilon u)$ of $P_\Sigma^{\lambda, \epsilon u}$, cf. (35), transforms as follows: The integral kernel of $L_\Sigma^{(S,\Lambda)} P_\Sigma^{\lambda, \epsilon u} (L_\Sigma^{(S,\Lambda)})^{-1}$ is given by

$$\begin{aligned} S p_\Sigma^{\lambda, \epsilon u}(\Lambda^{-1}x, \Lambda^{-1}y) S^* &= e^{-i\lambda(\Lambda^{-1}x, \Lambda^{-1}y)} S p_-(\Lambda^{-1}(y - x) + i\epsilon u) S^* \\ &= e^{-i\lambda(\Lambda^{-1}x, \Lambda^{-1}y)} p_-(y - x + i\epsilon \Lambda u) = p_{\Lambda\Sigma}^{\bar{\lambda}, \epsilon \Lambda u}(x, y), \end{aligned} \quad (52)$$

where $\bar{\lambda}(x, y) = \lambda(\Lambda^{-1}x, \Lambda^{-1}y)$. We claim $\bar{\lambda} \in \mathcal{G}(\Lambda A(\Lambda^{-1}\cdot))$. Indeed, $\bar{\lambda}$ clearly fulfills conditions (i)-(iii) of the Definition 2.2 of $\mathcal{G}(\Lambda A(\Lambda^{-1}\cdot))$. It also fulfills condition (iv) since

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \bar{\lambda}(x, y) \Big|_{y=x} &= \frac{\partial}{\partial x^\mu} \lambda(\Lambda^{-1}x, \Lambda^{-1}y) \Big|_{y=x} = (\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial z^\nu} \lambda(z, \Lambda^{-1}y) \Big|_{z=\Lambda^{-1}x, y=x} \\ &= \Lambda_\mu^\nu A_\nu(\Lambda^{-1}x) \end{aligned} \quad (53)$$

and similarly $\partial_\mu^y \bar{\lambda}(x, y) \Big|_{x=y} = -\Lambda_\mu^\nu A_\nu(\Lambda^{-1}x)$, where we have used $(\Lambda^{-1})^\nu_\mu = \Lambda_\mu^\nu$. This shows $L_\Sigma^{(S,\Lambda)} P_\Sigma^{\lambda, \epsilon u} (L_\Sigma^{(S,\Lambda)})^{-1} = P_\Sigma^{\bar{\lambda}, \epsilon \Lambda u}$, which implies $L_\Sigma^{(S,\Lambda)} P_\Sigma^\lambda (L_\Sigma^{(S,\Lambda)})^{-1} = P_\Sigma^{\bar{\lambda}}$ in the limit as $\epsilon \downarrow 0$; recall from Lemma 2.3 that the limit does not depend on the choice of $u, \Lambda u \in \text{Past}$.

Again by Lemma 2.9, there is a $W \in \mathcal{C}_{\Lambda\Sigma}(\Lambda A(\Lambda^{-1}\cdot))$ such that $P_{\Lambda\Sigma}^W - P_{\Lambda\Sigma}^{\bar{\lambda}} \in I_2(\mathcal{H}_{\Lambda\Sigma})$. We conclude

$$L_\Sigma^{(S,\Lambda)} P_\Sigma^V \left(L_\Sigma^{(S,\Lambda)} \right)^{-1} - P_{\Lambda\Sigma}^W = L_\Sigma^{(S,\Lambda)} \left(P_\Sigma^V - P_\Sigma^\lambda \right) \left(L_\Sigma^{(S,\Lambda)} \right)^{-1} - \left(P_{\Lambda\Sigma}^W - P_{\Lambda\Sigma}^{\bar{\lambda}} \right) \in I_2(\mathcal{H}_{\Lambda\Sigma}). \quad (54)$$

Claim (ii): The integral kernel of $e^{-i\Omega} P_\Sigma^{\lambda, \epsilon u} e^{i\Omega}$ for $\lambda \in \mathcal{G}(A)$, $\epsilon > 0$ and $u \in \text{Past}$ equals

$$e^{-i\Omega(x)} p_\Sigma^{\lambda, \epsilon u}(x, y) e^{i\Omega(y)} = e^{-i\Omega(x)} e^{-i\lambda(x, y)} p_-(y - x + i\epsilon u) e^{i\Omega(y)} = p_\Sigma^{\bar{\lambda}, \epsilon u}(x, y), \quad (55)$$

where $\bar{\lambda}(x, y) = \Omega(x) + \lambda(x, y) - \Omega(y)$, which clearly fulfills $\bar{\lambda} \in \mathcal{G}(A + \partial\Omega)$; cf. Definition 2.2. Taking the limit as $\epsilon \downarrow 0$, the claim follows from the same kind of reasoning as in part (i). \square

Finally, one can also use the self-adjoint operator P_Σ^A from (39) to construct a unitary operator $e^{Q_\Sigma^A} : \mathcal{H}_\Sigma \hookrightarrow$ which adapts the standard polarization \mathcal{H}_Σ^- to one corresponding to $A|_{T\Sigma}$, more precisely, $e^{Q_\Sigma^A} \mathcal{H}_\Sigma^- \in \mathcal{C}_\Sigma(A)$. It is defined as follows:

Definition 2.10. We set

$$Q_\Sigma^A := [P_\Sigma^A, P_\Sigma^-] = P_\Sigma^+(P_\Sigma^A - P_\Sigma^-)P_\Sigma^- - P_\Sigma^-(P_\Sigma^A - P_\Sigma^-)P_\Sigma^+. \quad (56)$$

Proof of Theorem 1.7. In this proof, we use a 2×2 -matrix notation for linear operators of the type $\mathcal{H}_\Sigma \hookrightarrow$. This matrix notation always refers to the splitting $\mathcal{H}_\Sigma = \mathcal{H}_\Sigma^+ \oplus \mathcal{H}_\Sigma^-$. In particular, we set

$$\begin{pmatrix} \Delta_{++} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{--} \end{pmatrix} = \Delta P_\Sigma^A = P_\Sigma^A - P_\Sigma^-, \quad (57)$$

cf. (36) for $\lambda = \lambda^A$. Using this matrix notation, we write

$$Q_\Sigma^A = \begin{pmatrix} 0 & \Delta_{+-} \\ -\Delta_{-+} & 0 \end{pmatrix}. \quad (58)$$

In the following we use the notation $X = Y \pmod{I_2(\mathcal{H}_\Sigma)}$ to mean $X - Y \in I_2(\mathcal{H}_\Sigma)$. By (iii) of Lemma 2.3 we know that $(\Delta P_\Sigma^A)^2 \in I_2(\mathcal{H}_\Sigma)$, and therefore

$$(P_\Sigma^A)^2 = (P_\Sigma^- + \Delta P_\Sigma^A)^2 = P_\Sigma^A + \begin{pmatrix} -\Delta_{++} & 0 \\ 0 & \Delta_{--} \end{pmatrix} \pmod{I_2(\mathcal{H}_\Sigma)}. \quad (59)$$

Furthermore, Lemma 2.9 implies for all $V \in \mathcal{C}_\Sigma(A)$ that the corresponding orthogonal projector P_Σ^V fulfills $P_\Sigma^A - P_\Sigma^V \in I_2(\mathcal{H}_\Sigma)$. However, this means also that $(P_\Sigma^A)^2 - P_\Sigma^A \in I_2(\mathcal{H}_\Sigma)$, and therefore, $\Delta_{++}, \Delta_{--} \in I_2(\mathcal{H}_\Sigma)$; see (59). In conclusion, we obtain

$$P_\Sigma^A = P_\Sigma^- + \Delta P_\Sigma^A = \begin{pmatrix} 0 & \Delta_{+-} \\ \Delta_{-+} & \text{id}_{\mathcal{H}_\Sigma^-} \end{pmatrix} \pmod{I_2(\mathcal{H}_\Sigma)}. \quad (60)$$

Since $(\Delta P_\Sigma^A)^2 \in I_2(\mathcal{H}_\Sigma)$ we have $\Delta_{-+}\Delta_{+-}, \Delta_{-+}\Delta_{+-} \in I_2(\mathcal{H}_\Sigma)$ and hence $(Q_\Sigma^A)^2 \in I_2(\mathcal{H}_\Sigma)$; cf. (58). Defining

$$\Pi_\Sigma^A := e^{Q_\Sigma^A} P_\Sigma^- e^{-Q_\Sigma^A}, \quad (61)$$

we conclude

$$\Pi_\Sigma^A = (\text{id}_{\mathcal{H}_\Sigma} + Q_\Sigma^A) P_\Sigma^- (\text{id}_{\mathcal{H}_\Sigma} - Q_\Sigma^A) = \begin{pmatrix} 0 & \Delta_{+-} \\ \Delta_{-+} & \text{id}_{\mathcal{H}_\Sigma^-} \end{pmatrix} = P_\Sigma^A = P_\Sigma^V \pmod{I_2(\mathcal{H}_\Sigma)}. \quad (62)$$

Furthermore, we observe that $e^{Q_\Sigma^A}$ is unitary because Q_Σ^A is skew-adjoint, so that Π_Σ^A is an orthogonal projector. Summarizing, we have shown $e^{Q_\Sigma^A} \mathcal{H}_\Sigma^- = \Pi_\Sigma^A \mathcal{H}_\Sigma \in \mathcal{C}_\Sigma(A)$, which proves the claim of Theorem 1.7.

As an addendum we prove the refinement of Theorem 1.7 described in Remark 1.8. For this it is left to show that $\text{charge}(U_{\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^-, \Pi_\Sigma^A \mathcal{H}_\Sigma) = 0$. We choose a future oriented foliation $(\Sigma_t)_{t \in \mathbb{R}}$ of space-time such that $\Sigma_0 = \Sigma_{\text{in}}$ and $\Sigma_1 = \Sigma$. Recall the choice of Σ_{in} described in (7). The operators $Q_{\Sigma_t}^A$ are compact because they are skew-adjoint and $(Q_{\Sigma_t}^A)^2 \in$

$I_2(\mathcal{H}_{\Sigma_t})$. Hence, the operators $e^{-Q_{\Sigma_t}^A}$ are compact perturbations of the identity operators $\text{id}_{\mathcal{H}_{\Sigma_t}}$. Translating this fact to an interaction picture, the operators

$$Q_t := U_{\Sigma_{\text{in}}\Sigma_t}^0 e^{-Q_{\Sigma_t}^A} U_{\Sigma_t\Sigma_{\text{in}}}^0 \quad (63)$$

are as well compact perturbations of the identity operator $\text{id}_{\mathcal{H}_{\Sigma_{\text{in}}}}$. We define the evolution operators in the interaction picture

$$U_t := U_{\Sigma_{\text{in}}\Sigma_t}^0 U_{\Sigma_t\Sigma_{\text{in}}}^A, \quad (64)$$

which are continuous in $t \in \mathbb{R}$ w.r.t. the operator norm; this follows from Lemma 3.9 in [3]. Moreover, using $V \approx W \Leftrightarrow P_{\Sigma}^V P_{\Sigma}^{W\perp}, P_{\Sigma}^{V\perp} P_{\Sigma}^W \in I_2(\mathcal{H}_{\Sigma})$, the just proven Theorem 1.7 implies

$$e^{Q_{\Sigma}^A} \mathcal{H}_{\Sigma}^{-} \approx U_{\Sigma\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^{-} \Rightarrow P_{\Sigma}^{\pm} e^{-Q_{\Sigma}^A} U_{\Sigma\Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^{\mp} \in I_2(\mathcal{H}_{\Sigma}) \quad (65)$$

$$\Rightarrow U_{\Sigma_{\text{in}}\Sigma}^0 P_{\Sigma}^{\pm} e^{-Q_{\Sigma}^A} U_{\Sigma\Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^{\mp} \in I_2(\mathcal{H}_{\Sigma}) \quad (66)$$

$$\Rightarrow P_{\Sigma_{\text{in}}}^{\pm} Q_t U_t P_{\Sigma_{\text{in}}}^{\mp} = P_{\Sigma}^{\pm} U_{\Sigma_{\text{in}}\Sigma}^0 e^{-Q_{\Sigma}^A} U_{\Sigma\Sigma_{\text{in}}}^A P_{\Sigma_{\text{in}}}^{\mp} \in I_2(\mathcal{H}_{\Sigma_{\text{in}}}). \quad (67)$$

Since $Q_t - \text{id}_{\mathcal{H}_{\Sigma_{\text{in}}}}$ is compact, the operator $P_{\Sigma_{\text{in}}}^{\pm} (Q_t - \text{id}_{\mathcal{H}_{\Sigma_{\text{in}}}}) U_t P_{\Sigma_{\text{in}}}^{\mp}$ is compact as well. Taking the difference with the compact operator in (67) yields that $P_{\Sigma_{\text{in}}}^{\pm} U_t P_{\Sigma_{\text{in}}}^{\mp}$ is compact so that

$$\begin{pmatrix} P_{\Sigma_{\text{in}}}^{+} U_t P_{\Sigma_{\text{in}}}^{+} & 0 \\ 0 & P_{\Sigma_{\text{in}}}^{-} U_t P_{\Sigma_{\text{in}}}^{-} \end{pmatrix} = U_t - \begin{pmatrix} 0 & P_{\Sigma_{\text{in}}}^{+} U_t P_{\Sigma_{\text{in}}}^{-} \\ P_{\Sigma_{\text{in}}}^{-} U_t P_{\Sigma_{\text{in}}}^{+} & 0 \end{pmatrix} \quad (68)$$

deviates from the unitary operator U_t by a compact perturbation, and hence, is a Fredholm operator. This implies that $P_{\Sigma_{\text{in}}}^{-} U_t P_{\Sigma_{\text{in}}}^{-}|_{\mathcal{H}_{\Sigma_{\text{in}}}^{-} \hookrightarrow \mathcal{H}_{\Sigma_{\text{in}}}^{-}}$ is a Fredholm operator. We note that the Fredholm index of $P_{\Sigma_{\text{in}}}^{-} U_{t=0} P_{\Sigma_{\text{in}}}^{-}|_{\mathcal{H}_{\Sigma_{\text{in}}}^{-} \hookrightarrow \mathcal{H}_{\Sigma_{\text{in}}}^{-}} = \text{id}_{\mathcal{H}_{\Sigma_{\text{in}}}^{-}}$ equals zero. The map $t \mapsto P_{\Sigma_{\text{in}}}^{-} U_t P_{\Sigma_{\text{in}}}^{-}$ is continuous in the operator norm which implies that the Fredholm index is constant, and hence,

$$\begin{aligned} 0 &= \text{index } P_{\Sigma_{\text{in}}}^{-} U_{t=1} P_{\Sigma_{\text{in}}}^{-}|_{\mathcal{H}_{\Sigma_{\text{in}}}^{-} \hookrightarrow \mathcal{H}_{\Sigma_{\text{in}}}^{-}} = \text{index } P_{\Sigma_{\text{in}}}^{-} U_{\Sigma_{\text{in}}\Sigma}^0 U_{\Sigma\Sigma_{\text{in}}}^A|_{\mathcal{H}_{\Sigma_{\text{in}}}^{-} \hookrightarrow \mathcal{H}_{\Sigma_{\text{in}}}^{-}} \\ &= \text{index } P_{\Sigma}^{-} U_{\Sigma\Sigma_{\text{in}}}^A|_{\mathcal{H}_{\Sigma_{\text{in}}}^{-} \rightarrow \mathcal{H}_{\Sigma}^{-}} = \text{index } P_{\Sigma}^{-}|_{U_{\Sigma\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^{-} \rightarrow \mathcal{H}_{\Sigma}^{-}} = \text{index } P_{\Sigma}^{-} e^{-Q_{\Sigma}^A}|_{U_{\Sigma\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^{-} \rightarrow \mathcal{H}_{\Sigma}^{-}} \\ &= \text{index } e^{Q_{\Sigma}^A} P_{\Sigma}^{-} e^{-Q_{\Sigma}^A}|_{U_{\Sigma\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^{-} \rightarrow \Pi_{\Sigma}^A \mathcal{H}_{\Sigma}} = \text{charge}(U_{\Sigma\Sigma_{\text{in}}}^A \mathcal{H}_{\Sigma_{\text{in}}}^{-}, \Pi_{\Sigma}^A \mathcal{H}_{\Sigma}), \end{aligned} \quad (69)$$

where in the fifth equality we have used that $e^{-Q_{\Sigma}^A}$ is a compact perturbation of the identity. \square

This concludes the proofs of the main results under the condition that the claims in Section 2.1 are true. The proofs of these claims will be provided in the next two sections.

2.3 Proof of Lemma 2.1, Lemma 2.3, and Theorem 2.4

Proof of Lemma 2.1. Given $\phi, \psi \in \mathcal{C}_{\Sigma}$, we set $\hat{\phi} = \mathcal{F}_{\mathcal{M}\Sigma} \phi$ and $\hat{\psi} = \mathcal{F}_{\mathcal{M}\Sigma} \psi$ where $\mathcal{F}_{\mathcal{M}\Sigma}$ is the generalized Fourier transform

$$(\mathcal{F}_{\mathcal{M}\Sigma} \psi)(p) = \frac{\not{p} + m}{2m} (2\pi)^{-3/2} \int_{\Sigma} e^{ipx} i_{\gamma}(d^4x) \psi(x) \quad \text{for } \psi \in \mathcal{C}_{\Sigma}, p \in \mathcal{M}, \quad (70)$$

introduced in Theorem 2.15 of [3]. This theorem ensures that $\widehat{\phi(p)}\widehat{\psi(p)}i_p(d^4p)$ is integrable on \mathcal{M}_- . Let $u \in \text{Past}$. With justifications given below, we compute the following.

$$\langle \phi, P_\Sigma^- \psi \rangle = \lim_{\epsilon \downarrow 0} \int_{p \in \mathcal{M}_-} e^{-\epsilon p u} \widehat{\phi(p)} \widehat{\psi(p)} \frac{i_p(d^4p)}{m} \quad (71)$$

$$= \frac{1}{(2\pi)^3 m} \lim_{\epsilon \downarrow 0} \int_{p \in \mathcal{M}_-} e^{-\epsilon p u} \int_{x \in \Sigma} \overline{\phi(x)} i_\gamma(d^4x) e^{-ipx} \left(\frac{\not{p} + m}{2m} \right)^2 \int_{y \in \Sigma} e^{ipy} i_\gamma(d^4y) \psi(y) i_p(d^4p) \quad (72)$$

$$= \frac{1}{(2\pi)^3 m} \lim_{\epsilon \downarrow 0} \int_{p \in \mathcal{M}_-} \int_{x \in \Sigma} \overline{\phi(x)} i_\gamma(d^4x) \frac{\not{p} + m}{2m} \int_{y \in \Sigma} e^{ip(y-x+i\epsilon u)} i_\gamma(d^4y) \psi(y) i_p(d^4p) \quad (73)$$

$$= \frac{1}{(2\pi)^3 m} \lim_{\epsilon \downarrow 0} \int_{x \in \Sigma} \overline{\phi(x)} i_\gamma(d^4x) \int_{y \in \Sigma} \int_{p \in \mathcal{M}_-} \frac{\not{p} + m}{2m} e^{ip(y-x+i\epsilon u)} i_p(d^4p) i_\gamma(d^4y) \psi(y) \quad (74)$$

$$= \lim_{\epsilon \downarrow 0} \int_{x \in \Sigma} \overline{\phi(x)} i_\gamma(d^4x) \int_{y \in \Sigma} p^-(y-x+i\epsilon u) i_\gamma(d^4y) \psi(y). \quad (75)$$

The interchange of the p -integral and the limit $\epsilon \downarrow 0$ in (71) is justified by dominated convergence since $\widehat{\phi(p)}\widehat{\psi(p)}i_p(d^4p)$ is integrable on \mathcal{M}_- and by $|e^{-\epsilon p u}| \leq 1$ for $\epsilon > 0$, $p \in \mathcal{M}_-$. In the step from (71) to (72) we have used (70) and that $\gamma^0(\gamma^\mu)^*\gamma^0 = \gamma^\mu$, from (72) to (73) that $\not{p}^2 = p^2$ and that $p^2 = m^2$ for $p \in \mathcal{M}_-$. In the step from (73) to (74) we have used Fubini's theorem to interchange the integrals. This is justified because ϕ and ψ are bounded and compactly supported, and because for any given $\epsilon > 0$, $|e^{ip(y-x+i\epsilon u)}| = e^{-\epsilon p u}$ tends exponentially fast to 0 as $|p| \rightarrow \infty$, $p \in \mathcal{M}_-$. This proves the claim (29).

Now we prove the claimed properties of D and p^- . For any $w \in \mathbb{R}^4 + i \text{Past}$, the modulus $|e^{ipw}| = e^{-p \text{Im } w}$ tends exponentially fast to 0 as $|p| \rightarrow \infty$, $p \in \mathcal{M}_-$. Consequently, exchanging differentiation and integration in the following calculation is justified:

$$\begin{aligned} p^-(w) &= \frac{1}{(2\pi)^3 m} \int_{\mathcal{M}_-} \frac{-i\not{\partial}^w + m}{2m} e^{ipw} i_p(d^4p) \\ &= \frac{1}{(2\pi)^3 m} \frac{-i\not{\partial}^w + m}{2m} \int_{\mathcal{M}_-} e^{ipw} i_p(d^4p) = \frac{-i\not{\partial}^w + m}{2m} D(w). \end{aligned} \quad (76)$$

To show the second equality in (31), we proceed as follows: First, we show that $w \in \mathbb{R}^4 + i \text{Past}$ implies $-w_\mu w^\mu \in \mathbb{C} \setminus \mathbb{R}_0^- = \text{domain}(\sqrt{\cdot})$. We take $w = z + iu$ with $z \in \mathbb{R}^4$ and $u \in \text{Past}$, and assume $-w_\mu w^\mu \in \mathbb{R}$. Then $0 = \text{Im}(w_\mu w^\mu) = 2z_\mu u^\mu$, i.e., z is orthogonal to u in the Minkowski sense. Because u is time-like, we conclude that z is space-like or zero. We obtain $w_\mu w^\mu = \text{Re}(w_\mu w^\mu) = z_\mu z^\mu - u_\mu u^\mu < 0$, i.e., $-w_\mu w^\mu \in \text{domain}(\sqrt{\cdot})$. It follows that $\sqrt{-w_\mu w^\mu} \in \mathbb{R}^+ + i\mathbb{R} = \text{domain}(K_1)$. In particular,

$$\tilde{D} : \mathbb{R}^4 + i \text{Past} \ni w \mapsto -\frac{m^3}{2\pi^2} \frac{K_1(m\sqrt{-w_\mu w^\mu})}{m\sqrt{-w_\mu w^\mu}} \quad (77)$$

is a well-defined holomorphic function. Because $|e^{ipw}|$ decays fast as $|p| \rightarrow \infty$, $p \in \mathcal{M}_-$, uniformly for w in any compact subset of $\mathbb{R}^4 + i \text{Past}$,

$$D : \mathbb{R}^4 + i \text{Past} \ni w \mapsto \frac{1}{(2\pi)^3 m} \int_{\mathcal{M}_-} e^{ipw} i_p(d^4 p) \quad (78)$$

is also a holomorphic function. We need to show $D = \tilde{D}$. By the identity theorem for holomorphic functions, it suffices to show that the restrictions of D and \tilde{D} to $i \text{Past}$ coincide. Given $w = iu \in i \text{Past}$, we choose a proper, orthochronous Lorentz transform $\Lambda \in \text{SO}^\uparrow(1, 3) \subseteq \mathbb{R}^{4 \times 4}$ that maps u to the negative time axis:

$$\Lambda u = -te_0 = (-t, 0, 0, 0) \text{ with } t = \sqrt{u_\mu u^\mu} = \sqrt{-w_\mu w^\mu} > 0. \quad (79)$$

By Lorentz invariance of the volume-form $i_p(d^4 p)$ on \mathcal{M}_- , we know

$$\int_{\mathcal{M}_-} e^{ipw} i_p(d^4 p) = \int_{\mathcal{M}_-} e^{ip\Lambda w} i_p(d^4 p) \quad (80)$$

and $\sqrt{-w_\mu w^\mu} = \sqrt{-(\Lambda w)_\mu (\Lambda w)^\mu}$. Summarizing, we have reduced the claim $D = \tilde{D}$ to its special case $D(w) = \tilde{D}(w)$ for $w = -ite_0$, $t = \sqrt{-w_\mu w^\mu} > 0$. This special case is proven as follows. Using

$$i_p(d^4 p) = \frac{m^2}{p^0} d^3 p \text{ on } (T_p \mathcal{M})^3, \quad (81)$$

rotational symmetry, and the substitution

$$s = \frac{\sqrt{k^2 + m^2}}{m}, \quad k = m\sqrt{s^2 - 1}, \quad m^2 s ds = k dk, \quad (82)$$

we obtain with the abbreviation $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$:

$$\begin{aligned} \int_{\mathcal{M}_-} e^{ipw} i_p(d^4 p) &= -m^2 \int_{\mathbb{R}^3} e^{-E(\mathbf{p})t} \frac{d^3 \mathbf{p}}{E(\mathbf{p})} \\ &= -4\pi m^2 \int_0^\infty \exp\left(-t\sqrt{k^2 + m^2}\right) \frac{k^2 dk}{\sqrt{k^2 + m^2}} \\ &= -4\pi m^4 \int_1^\infty e^{-mts} \sqrt{s^2 - 1} ds = -4\pi m^4 \frac{K_1(mt)}{mt}, \end{aligned} \quad (83)$$

using the definition of K_1 in (32), and hence, the claim $D(-ite_0) = \tilde{D}(-ite_0)$.

The representation (77) of D shows also that D can be analytically extended to all arguments $w \in \mathbb{C}^4$ with $-w_\mu w^\mu \in \text{domain}(\sqrt{\cdot}) = \mathbb{C} \setminus \mathbb{R}_0^-$. The same holds true for $p^- = (2m)^{-1}(-i\cancel{\partial} + m)D$. To sum up, p^- has an analytic continuation $p^- : \text{domain}(r) \rightarrow \mathbb{C}^{4 \times 4}$, which also concludes the proof of Lemma 2.1. \square

Proof of Lemma 2.3. We remark that most of the arguments in this proof are valid without regularization, i.e., also in the case $\epsilon = 0$. This is in contrast to Section 2.4 below, where the regularization with $\epsilon > 0$ turns out to be very useful.

Let $A \in \mathcal{C}_c^\infty(\mathbb{R}^4, \mathbb{R}^4)$, $\lambda \in \mathcal{G}(A)$, and Σ be a Cauchy surface. Before proving the claim (34)-(35) it will be convenient to introduce the operators $\Delta P_\Sigma^{\lambda, \epsilon u}$, $\epsilon \geq 0$, which shall act on any $\psi \in \mathcal{H}_\Sigma$ as

$$\left(\Delta P_\Sigma^{\lambda, \epsilon u} \psi \right) (x) = \int_{y \in \Sigma} (e^{-i\lambda(x, y)} - 1) p^-(y - x + i\epsilon u) i_\gamma(d^4 y) \psi(y), \quad (84)$$

where the fixed vector $u \in \mathbb{R}^4$ is past-directed time-like. We remark that the special case $\epsilon = 0$ is included in the form $\Delta P_\Sigma^{\lambda, 0} = \Delta P_\Sigma^\lambda$; cf. (36).

We show now that $\Delta P_\Sigma^{\lambda, \epsilon u} : \mathcal{H}_\Sigma \hookrightarrow \mathcal{H}_\Sigma$ is well-defined. Recall the parametrization $\pi_\Sigma(\mathbf{x})$ of Σ as stated in (13) and the identity $i_\gamma(d^4 x) = \Gamma(\mathbf{x}) d^3 x$ on $(T_x \Sigma)^3$ given in (19). We use the abbreviation $x = \pi_\Sigma(\mathbf{x})$, $y = \pi_\Sigma(\mathbf{y})$ in the following. Line (84) can be recast into

$$\left(\Delta P_\Sigma^{\lambda, \epsilon u} \psi \right) (x) = \int_{\mathbb{R}^3} \Delta p_\Sigma^{\lambda, \epsilon u}(\mathbf{x}, \mathbf{y}) \Gamma(\mathbf{y}) \psi(y) d^3 \mathbf{y} \quad \text{for} \quad (85)$$

$$\Delta p_\Sigma^{\lambda, \epsilon u}(\mathbf{x}, \mathbf{y}) := (e^{-i\lambda(x, y)} - 1) p^-(y - x + i\epsilon u). \quad (86)$$

To show at the same time that the right-hand side of (85), i.e., (84), is well-defined for $\psi \in \mathcal{H}_\Sigma$ and almost every $x \in \Sigma$, and that $\Delta P_\Sigma^{\lambda, \epsilon} \psi \in \mathcal{H}_\Sigma$, it suffices to prove that for every $\phi \in \mathcal{H}_\Sigma$, we have

$$\int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} \left| \overline{\phi(x)} \Gamma(\mathbf{x}) \Delta p_\Sigma^{\lambda, \epsilon u}(\mathbf{x}, \mathbf{y}) \Gamma(\mathbf{y}) \psi(y) \right| d^3 \mathbf{y} \leq C_2 \|\phi\| \|\psi\| \quad (87)$$

with some constant $C_2(u, V_{\max})$. We collect the necessary ingredients:

- As λ is smooth and vanishes on the diagonal, there is a positive constant $C_3(\lambda)$ such that

$$|e^{-i\lambda(x, y)} - 1| \leq C_3 |x - y| [1_K(x) \vee 1_K(y)] \text{ for } x, y \in \mathbb{R}^4. \quad (88)$$

Note that this bound holds globally, not only locally close to the diagonal, because $e^{-i\lambda} - 1$ is bounded and vanishes outside $K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$ for some compact set K .

- The bounds (28) from the appendix, cf. (15), show that for all $x, y \in \Sigma$ and $(z^0, \mathbf{z}) = z = y - x$ we find $|\mathbf{z}| \leq |z| \leq \sqrt{1 + V_{\max}^2} |\mathbf{z}|$.
- Formula (238) in Corollary A.1 of the Appendix ensures for all $\epsilon \geq 0$ that for all $z = (z^0, \mathbf{z})$ such that $z = y - x$ for $x, y \in \Sigma$ and $\mathbf{z} \neq 0$ that

$$\|p^-(z + i\epsilon u)\| \leq O_{u, V_{\max}} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^3} \right). \quad (89)$$

Thanks to these ingredients we find the estimate

$$\|\Delta p_{\Sigma}^{\lambda, \epsilon u}(\mathbf{x}, \mathbf{y})\| \leq C_4 \frac{e^{-C_D |\mathbf{y} - \mathbf{x}|}}{|\mathbf{y} - \mathbf{x}|^2} [1_K(x) \vee 1_K(y)] \quad (90)$$

for all $x, y \in \Sigma$ such that $\mathbf{y} - \mathbf{x} \neq 0$ and $\epsilon \geq 0$ with some constant $C_4(u, V_{\max}, \lambda)$. Consequently, using the bound for Γ from (20), we have the dominating function

$$\sup_{\epsilon \geq 0} \left| \overline{\phi(x)} \Gamma(\mathbf{x}) \Delta p_{\Sigma}^{\lambda, \epsilon u}(\mathbf{x}, \mathbf{y}) \Gamma(\mathbf{y}) \psi(y) \right| \leq C_4 \Gamma_{\max}^2 |\phi(x)| \frac{e^{-C_D |\mathbf{y} - \mathbf{x}|}}{|\mathbf{y} - \mathbf{x}|^2} |\psi(y)|, \quad (91)$$

which is integrable, as the following calculation shows:

$$C_4 \Gamma_{\max}^2 \int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} |\phi(x)| \frac{e^{-C_D |\mathbf{y} - \mathbf{x}|}}{|\mathbf{y} - \mathbf{x}|^2} |\psi(y)| d^3 \mathbf{y} d^3 \mathbf{x} \quad (92)$$

$$= C_4 \Gamma_{\max}^2 \int_{\mathbf{z} \in \mathbb{R}^3} \frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^2} \int_{\mathbf{x} \in \mathbb{R}^3} |\phi(\pi_{\Sigma}(\mathbf{x}))| |\psi(\pi_{\Sigma}(\mathbf{x} + \mathbf{z}))| d^3 \mathbf{x} d^3 \mathbf{z} \quad (93)$$

$$\leq 4\pi C_4 \Gamma_{\max}^2 \int_0^{\infty} e^{-C_D s} ds \|\phi \circ \pi_{\Sigma}\|_2 \|\psi \circ \pi_{\Sigma}\|_2 \quad (94)$$

$$\leq C_2 \|\phi\| \|\psi\|, \quad (95)$$

for a constant $C_2(u, V_{\max}, \lambda)$. In the step from (93) to (94) we use the Cauchy-Schwarz inequality, and in the step from (94) to (95), we use that the norms $\|\cdot \circ \pi_{\Sigma}\|_2$ and $\|\cdot\|$ are equivalent. On the one hand, this proves claim (87), which implies that the operators $\Delta P_{\Sigma}^{\lambda, \epsilon u} : \mathcal{H}_{\Sigma} \hookrightarrow$ described in (85) and (86) are well-defined for all $\epsilon \geq 0$ and bounded by

$$\sup_{\epsilon \geq 0} \|\Delta P_{\Sigma}^{\lambda, \epsilon u}\|_{\mathcal{H}_{\Sigma} \hookrightarrow} \leq C_2. \quad (96)$$

On the other hand, we use again the integrable domination from (91) together with the point-wise convergence

$$\lim_{\epsilon \downarrow 0} p^-(y - x + i\epsilon u) = p^-(y - x) \quad (97)$$

for $x, y \in \Sigma$ with $x \neq y$; cf. the analytic continuation of p^- described in Lemma 2.1. Using these ingredients, the dominated convergence theorem yields the following convergence in the weak operator topology:

$$\left\langle \phi, \Delta P_{\Sigma}^{\lambda, \epsilon u} \psi \right\rangle \xrightarrow{\epsilon \downarrow 0} \left\langle \phi, \Delta P_{\Sigma}^{\lambda} \psi \right\rangle \text{ for } \phi, \psi \in \mathcal{H}_{\Sigma}. \quad (98)$$

The next argument needs this fact only restricted to $\phi, \psi \in \mathcal{C}_{\Sigma}$. Using the notation (35) and Lemma 2.1, we get for $\phi, \psi \in \mathcal{C}_{\Sigma}$

$$\left\langle \phi, P_{\Sigma}^{\lambda, \epsilon u} \psi \right\rangle = \left\langle \phi, P_{\Sigma}^{0, \epsilon u} \psi \right\rangle + \left\langle \phi, \Delta P_{\Sigma}^{\lambda, \epsilon u} \psi \right\rangle \xrightarrow{\epsilon \downarrow 0} \left\langle \phi, P_{\Sigma}^{-} \psi \right\rangle + \left\langle \phi, \Delta P_{\Sigma}^{\lambda} \psi \right\rangle. \quad (99)$$

Because $P_{\Sigma}^{-}, \Delta P_{\Sigma}^{\lambda} : \mathcal{H}_{\Sigma} \hookrightarrow$ are bounded operators and \mathcal{C}_{Σ} is dense in \mathcal{H}_{Σ} , this implies that

$$P_{\Sigma}^{\lambda} := P_{\Sigma}^{-} + \Delta P_{\Sigma}^{\lambda} : \mathcal{H}_{\Sigma} \hookrightarrow \quad (100)$$

is the unique bounded operator that satisfies (34), together with the bound

$$\|P_\Sigma^\lambda\|_{\mathcal{H}_\Sigma \hookrightarrow} \leq \|P_\Sigma^-\|_{\mathcal{H}_\Sigma \hookrightarrow} + \|\Delta P_\Sigma^\lambda\|_{\mathcal{H}_\Sigma \hookrightarrow} \leq 1 + C_2(u, V_{\max}, \lambda) \quad (101)$$

coming from (96). Note that we may take any fixed $u \in \text{Past}$, e.g., $u = (-1, 0, 0, 0)$, in this bound and in the bounds below.

Next, we show that $K^\lambda := |\Delta P_\Sigma^\lambda|^2$ is a Hilbert-Schmidt operator. It is the integral operator (here written in 3-vector notation)

$$K^\lambda \psi(x) = \int_{\mathbb{R}^3} k^\lambda(\mathbf{x}, \mathbf{y}) \Gamma(\mathbf{y}) \psi(y) d^3 \mathbf{y} \quad (102)$$

for $\psi \in \mathcal{H}_\Sigma$ and almost all $x \in \Sigma$ with the integral kernel

$$k^\lambda(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^3} \gamma^0 \Delta p_\Sigma^{\lambda,0}(\mathbf{x}, \mathbf{z})^* \gamma^0 \Gamma(\mathbf{z}) \Delta p_\Sigma^{\lambda,0}(\mathbf{z}, \mathbf{y}) d^3 \mathbf{z}. \quad (103)$$

We remark that under the symmetry assumption $\lambda(x, y) = -\lambda(y, x)$, we have

$$\gamma^0 \Delta p_\Sigma^{\lambda,0}(\mathbf{x}, \mathbf{z})^* \gamma^0 = \Delta p_\Sigma^{\lambda,0}(\mathbf{z}, \mathbf{x}); \quad (104)$$

cf. formula (110) below. Thanks to the estimate (90) we find

$$\|k^\lambda(\mathbf{x}, \mathbf{y})\| \leq \Gamma_{\max} C_4^2 \int_{\mathbb{R}^3} \frac{e^{-C_D|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|^2} \frac{e^{-C_D|\mathbf{z}-\mathbf{y}|}}{|\mathbf{z}-\mathbf{y}|^2} (1_K(x) \vee 1_K(z))(1_K(z) \vee 1_K(y)) d^3 \mathbf{z}. \quad (105)$$

Next, we use the bound

$$e^{-C_D|\mathbf{x}-\mathbf{z}|} e^{-C_D|\mathbf{z}-\mathbf{y}|} (1_K(x) \vee 1_K(z))(1_K(z) \vee 1_K(y)) \leq C_5 e^{-C_D(|\mathbf{y}-\mathbf{x}|+|\mathbf{x}|)/2} \quad (106)$$

with the constant $C_5(\lambda, V_{\max}) = \sup_{z \in K} e^{C_D|z|/2}$. Substituting this bound in (105) and carrying out the integration yields

$$\|k^\lambda(\mathbf{x}, \mathbf{y})\| \leq \Gamma_{\max} C_4^2 C_5 e^{-C_D(|\mathbf{y}-\mathbf{x}|+|\mathbf{x}|)/2} \int_{\mathbb{R}^3} \frac{d^3 \mathbf{z}}{|\mathbf{x}-\mathbf{z}|^2 |\mathbf{z}-\mathbf{y}|^2} = C_6 \frac{e^{-C_D(|\mathbf{y}-\mathbf{x}|+|\mathbf{x}|)/2}}{|\mathbf{y}-\mathbf{x}|} \quad (107)$$

for a finite constant $C_6(\lambda, V_{\max})$. We can therefore bound the Hilbert-Schmidt norm of K^λ as follows:

$$\begin{aligned} \|K^\lambda\|_{I_2(\mathcal{H}_\Sigma)}^2 &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \text{trace}[\gamma^0 k^\lambda(\mathbf{x}, \mathbf{y})^* \gamma^0 \Gamma(\mathbf{x}) k^\lambda(\mathbf{x}, \mathbf{y}) \Gamma(\mathbf{y})] d^3 \mathbf{x} d^3 \mathbf{y} \\ &\leq 4\Gamma_{\max}^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|k^\lambda(\mathbf{x}, \mathbf{y})\|^2 d^3 \mathbf{x} d^3 \mathbf{y} \\ &\leq 4\Gamma_{\max}^2 C_6^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{-C_D(|\mathbf{y}-\mathbf{x}|+|\mathbf{x}|)}}{|\mathbf{y}-\mathbf{x}|^2} d^3 \mathbf{x} d^3 \mathbf{y} < \infty. \end{aligned} \quad (108)$$

This proves that $K^\lambda = |\Delta P_\Sigma^\lambda|^2$ is a Hilbert-Schmidt operator, and therefore, ΔP_Σ^λ is compact.

To prove part (iv) of Lemma 2.3, we assume $\lambda(x, y) = -\lambda(y, x)$ for all $x, y \in \Sigma$. From the symmetries $D(w^*) = D(w)^*$ and $D(-w) = D(w)$ for all $w \in \text{domain}(r)$ and $(\gamma^\mu)^* = \gamma^0 \gamma^\mu \gamma^0$, we conclude

$$p^-(-w^*) = \gamma^0 p^-(w) \gamma^0, \quad (109)$$

and hence, using the assumed symmetry of λ ,

$$\gamma^0 (e^{-i\lambda(y, x)} p_-(y - x + i\epsilon u))^* \gamma^0 = e^{-i\lambda(x, y)} p_-(x - y + i\epsilon u) \quad (110)$$

for $x, y \in \Sigma$, $\epsilon > 0$ and $u \in \text{Past}$. Substituting this in the specification (34)-(35) of P_Σ^λ , it follows that P_Σ^λ is self-adjoint and concludes the proof. \square

Proof of Theorem 2.4. To show the equivalence we need to control of the kernel of $P_\Sigma^\lambda - P_\Sigma^{\tilde{\lambda}}$ from above and from below. Let $\Delta \mathbf{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the vector field on \mathbb{R}^3 with

$$\Delta \mathbf{A}(\mathbf{x}) \cdot \mathbf{z} = (A_\mu(x) - \tilde{A}_\mu(x)) z^\mu \quad (111)$$

for any $x = (x^0, \mathbf{x}) \in \Sigma$ and $z = (z^0, \mathbf{z}) \in T_x \Sigma$. Then for any $x = (x^0, \mathbf{x}) \in \Sigma$, $A(x)|_{T_x \Sigma} = \tilde{A}(x)|_{T_x \Sigma}$ holds if and only if $\Delta \mathbf{A}(\mathbf{x}) = 0$. From $\lambda \in \mathcal{G}(A)$ and $\tilde{\lambda} \in \mathcal{G}(\tilde{A})$, see Definition 2.2, we get the Taylor expansions

$$e^{-i\lambda(x, y)} = 1 + iA_\mu(x)(y^\mu - x^\mu) + O_\lambda(|x - y|^2)(1_K(x) \vee 1_K(y)), \quad (112)$$

$$e^{-i\tilde{\lambda}(x, y)} = 1 + i\tilde{A}_\mu(x)(y^\mu - x^\mu) + O_{\tilde{\lambda}}(|x - y|^2)(1_K(x) \vee 1_K(y)), \quad (113)$$

$$y^0 - x^0 = \nabla t_\Sigma(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + O_\Sigma(|\mathbf{x} - \mathbf{y}|^2) \quad (114)$$

for $y, x \in \Sigma$ from which we conclude

$$e^{-i\lambda(x, y)} - e^{-i\tilde{\lambda}(x, y)} = i\Delta \mathbf{A}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + r_1(\mathbf{x}, \mathbf{y}) \quad (115)$$

with an error term r_1 that fulfills for any $x, y \in \Sigma$

$$|r_1(\mathbf{x}, \mathbf{y})| \leq O_{\lambda, \tilde{\lambda}, V_{\max}}(|\mathbf{x} - \mathbf{y}|^2)(1_K(x) \vee 1_K(y)), \quad (116)$$

where we used $|x - y| = O_{V_{\max}}(|\mathbf{x} - \mathbf{y}|)$ due to (15). Note that the bound (116) holds not only locally near the diagonal but also *globally* for $x, y \in \Sigma$ because $e^{-i\lambda} - e^{-i\tilde{\lambda}}$ is bounded and λ and $\tilde{\lambda}$ vanish outside $K \times \mathbb{R}^4 \cup \mathbb{R}^4 \times K$ for some compact set $K \subset \mathbb{R}^4$. For $\phi, \psi \in \mathcal{H}_\Sigma$ formula (36) from Lemma 2.3 implies

$$\begin{aligned} & \langle \phi, (P_\Sigma^\lambda - P_\Sigma^{\tilde{\lambda}}) \psi \rangle \\ &= \int_{x \in \Sigma} \bar{\phi}(x) i_\gamma(d^4 x) \int_{y \in \Sigma} (e^{-i\lambda(x, y)} - e^{-i\tilde{\lambda}(x, y)}) p^-(y - x) i_\gamma(d^4 y) \psi(y) \\ &= \int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} \phi(x)^* \gamma^0 \Gamma(\mathbf{x}) [t_1(x, y) + t_2(x, y)] \gamma^0 \Gamma(\mathbf{y}) \psi(y) d^3 \mathbf{y} d^3 \mathbf{x} \end{aligned} \quad (117)$$

with

$$t_1(x, y) = i\Delta\mathbf{A}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})p^-(y - x)\gamma^0, \quad (118)$$

$$t_2(x, y) = r_1(\mathbf{x}, \mathbf{y})p^-(y - x)\gamma^0, \quad (119)$$

where we use the abbreviations $x = \pi_\Sigma(\mathbf{x})$, $y = \pi_\Sigma(\mathbf{y})$ again, and Γ is defined in (19). We have introduced two extra factors γ^0 in (117) in order to have a positive-definite weight $\gamma^0\Gamma$.

We claim that the kernel $t_2(x, y)\gamma^0$ gives rise to a Hilbert-Schmidt-operator T_2 . Indeed, using the bound (20) for Γ , the bound (238) from Corollary A.1 in the appendix for p^- , and the bound (116) for r_1 , we have

$$\begin{aligned} \|T_2\|_{L_2(\mathcal{H}_\Sigma)}^2 &= \int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} \text{trace}[t_2(x, y)^* \gamma^0 \Gamma(\mathbf{x}) t_2(x, y) \gamma^0 \Gamma(\mathbf{y})] d^3\mathbf{y} d^3\mathbf{x} \\ &\leq C_7 \int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} \left| \frac{e^{-C_D|\mathbf{y}-\mathbf{x}|^2}}{|\mathbf{y} - \mathbf{x}|} \right|^2 (1_K(x) + 1_K(y)) d^3\mathbf{y} d^3\mathbf{x} \leq C_8 < \infty \end{aligned} \quad (120)$$

for some constants C_7 and C_8 that depend on $\Sigma, \lambda, \tilde{\lambda}$.

If $A|_{T\Sigma} = \tilde{A}|_{T\Sigma}$ then $\Delta\mathbf{A} = 0$. This implies $t_1 = 0$ and therefore $P_\Sigma^\lambda - P_\Sigma^{\tilde{\lambda}} = T_2$ is a Hilbert-Schmidt operator. This proves the “ \Leftarrow ” part of the claim (40).

Conversely, let us assume that $A|_{T\Sigma} = \tilde{A}|_{T\Sigma}$ does not hold. Then we can take some $x_0 \in \mathbb{R}^3$ with $\Delta\mathbf{A}(\mathbf{x}_0) \neq 0$. By continuity of $\Delta\mathbf{A}$, we have $\inf_{\mathbf{x} \in U} |\Delta\mathbf{A}(\mathbf{x})| > 0$ for some neighborhood U of \mathbf{x} . Furthermore there is a constant $C_9(V_{\max})$ such that $\gamma^0\Gamma(\mathbf{x}) - C_9$ is positive-semidefinite for all $x = (x^0, \mathbf{x}) \in \Sigma$. Consequently, we get the following bound for all $\mathbf{x} \in U$ and $\mathbf{y} \in \mathbb{R}^3$:

$$\begin{aligned} \text{trace} \left[t_1(\mathbf{x}, \mathbf{y})^* \gamma^0 \Gamma(\mathbf{x}) t_1(\mathbf{x}, \mathbf{y}) \gamma^0 \Gamma(\mathbf{y}) \right] &\geq C_9^2 \text{trace} \left[t_1(\mathbf{x}, \mathbf{y})^* t_1(\mathbf{x}, \mathbf{y}) \right] \\ &\geq C_{10} |\Delta\mathbf{A}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})|^2 \|p^-(y - x)\|^2 \geq C_{11} |\Delta\mathbf{A}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})|^2 \left(\frac{e^{-m|\mathbf{y}-\mathbf{x}|}}{|\mathbf{y} - \mathbf{x}|^3} \right)^2. \end{aligned} \quad (121)$$

with two positive constants C_{10} and C_{11} depending on V_{\max} . In the last step, we have used the lower bound (239) for $\|p^-\|$ from Corollary A.2 in the appendix. Because the lower bound given in (121) is not integrable over $(\mathbf{x}, \mathbf{y}) \in U \times \mathbb{R}^4$, we conclude that T_1 is not a Hilbert-Schmidt operator. Because T_2 is a Hilbert-Schmidt operator, this implies that $P_\Sigma^\lambda - P_\Sigma^{\tilde{\lambda}}$ cannot be a Hilbert-Schmidt operator. Thus, we have proven part “ \Rightarrow ” of the Theorem. \square

2.4 Proof of Theorem 2.8

This section contains the centerpiece of this work. The proof of Theorem 2.8 will be given at the end of this section. To show that the claimed equality (47) holds, we analyze the difference of matrix elements

$$\left\langle \phi, (P_{\Sigma_{t_1}}^A + S_{\Sigma_{t_1}}^A) \psi \right\rangle - \left\langle \phi, (P_{\Sigma_{t_0}}^A + S_{\Sigma_{t_0}}^A) \psi \right\rangle \quad (122)$$

for $\psi, \phi \in \mathcal{C}_A$. This is done in two steps. First, using Stokes' theorem, we provide a formula for the derivative w.r.t. the flow parameter of the family of Cauchy surfaces $(\Sigma_t)_{t \in T}$ in Lemma 2.11 and Corollary 2.12. Second, we give the relevant estimates on this derivative in Lemmas 2.13-2.15 which are summarized in Corollary 2.16, and conclude with the proof of Theorem 2.8.

For the first step, the following notations for the Dirac operators acting from the left and from the right, respectively, are convenient:

$$D^A \psi(x) = D_x^A \psi(x) := (i\overleftarrow{\partial}^x - A(x) - m)\psi(x), \quad (123)$$

$$\overline{\phi(y)} \overleftarrow{D}^A = \overline{\phi(y)} \overleftarrow{D}_y^A := \overline{\phi(y)}(-i\overleftarrow{\partial}^y - A(y) - m) = \overline{D_y^A \phi(y)}, \quad (124)$$

where $f(y)\overleftarrow{\partial}^y = f(y)\overleftarrow{\partial} := \partial_\mu f(y)\gamma^\mu$.

Lemma 2.11. *Let $k : \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{C}^{4 \times 4}$ be a smooth function. Let $\phi, \psi \in \mathcal{C}_A$. Then for any $t \in T$ we have*

$$\begin{aligned} & \frac{d}{dt} \int_{x \in \Sigma_t} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4 x) k(x, y) i_\gamma(d^4 y) \psi(y) \\ &= -i \int_{x \in \Sigma_t} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4 x) \mathcal{D}_t^A k(x, y) i_\gamma(d^4 y) \psi(y) \end{aligned} \quad (125)$$

with

$$\mathcal{D}_t^A k(x, y) := v_t(x) \not{n}_t(x) D_x^A k(x, y) - k(x, y) \overleftarrow{D}_y^A v_t(y) \not{n}_t(y). \quad (126)$$

Proof. Assume that $\phi', \psi' : \mathbb{R}^4 \rightarrow \mathbb{C}^4$ are smooth functions with $\text{supp } \phi' \cap \text{supp } \psi' \subseteq K + \text{Causal}$ for some compact set $K \subset \mathbb{R}^4$.

We set

$$\Sigma_{t_0 t_1} := \{(x, t) \in \Sigma \mid t_0 \leq t \leq t_1\} \quad (127)$$

for any real numbers $t_0 \leq t_1$. By Stokes' theorem, we have:

$$\left(\int_{\Sigma_{t_1}} - \int_{\Sigma_{t_0}} \right) \overline{\phi'(x)} i_\gamma(d^4 x) \psi'(x) = \int_{\Sigma_{t_0 t_1}} d[\overline{\phi'(x)} i_\gamma(d^4 x) \psi'(x)]. \quad (128)$$

We calculate:

$$\begin{aligned} d[\overline{\phi'(x)} i_\gamma(d^4 x) \psi'(x)] &= \partial_\mu (\overline{\phi'(x)} \gamma^\mu \psi'(x)) d^4 x \\ &= (\partial_\mu \overline{\phi'(x)}) \gamma^\mu \psi'(x) d^4 x + \overline{\phi'(x)} \gamma^\mu \partial_\mu \psi'(x) d^4 x \\ &= \overleftarrow{\partial} \phi'(x) \psi'(x) d^4 x + \phi'(x) \overleftarrow{\partial} \psi'(x) d^4 x \\ &= i \overleftarrow{D}^A \phi'(x) \psi'(x) d^4 x - i \overline{\phi'(x)} D^A \psi'(x) d^4 x, \end{aligned} \quad (129)$$

see also the calculation from (17) to (20) in [3]. Integration yields

$$\begin{aligned}
& \left(\int_{\Sigma_{t_1}} - \int_{\Sigma_{t_0}} \right) \overline{\phi'(x)} i_\gamma(d^4x) \psi'(x) \\
&= i \int_{\Sigma_{t_0 t_1}} [\overline{D^A \phi'(x)} \psi'(x) - \overline{\phi'(x)} D^A \psi'(x)] d^4x \\
&= i \int_{t_0}^{t_1} \int_{\Sigma_t} [\overline{D^A \phi'(x)} \psi'(x) - \overline{\phi'(x)} D^A \psi'(x)] i_{v_t n_t}(d^4x) dt.
\end{aligned} \tag{130}$$

Differentiating this with respect to the upper boundary t_1 , we conclude

$$\begin{aligned}
& \frac{d}{dt} \int_{\Sigma_t} \overline{\phi'(x)} i_\gamma(d^4x) \psi'(x) \\
&= i \int_{\Sigma_t} [\overline{D^A \phi'(x)} \psi'(x) - \overline{\phi'(x)} D^A \psi'(x)] i_{v_t n_t}(d^4x) \\
&= i \int_{\Sigma_t} [\overline{\phi'(x)} \overleftarrow{D^A} v_t(x) \not{n}_t(x) i_\gamma(d^4x) \psi'(x) - \overline{\phi'(x)} i_\gamma(d^4x) v_t(x) \not{n}_t(x) D^A \psi'(x)],
\end{aligned} \tag{131}$$

using (19). In the special case $\phi' \in \mathcal{C}_A$ this boils down to

$$\frac{d}{dt} \int_{\Sigma_t} \overline{\phi'(x)} i_\gamma(d^4x) \psi'(x) = -i \int_{\Sigma_t} \overline{\phi'(x)} i_\gamma(d^4x) v_t(x) \not{n}_t(x) D^A \psi'(x), \tag{132}$$

while in the special case $\psi' \in \mathcal{C}_A$ it boils down to

$$\frac{d}{dt} \int_{\Sigma_t} \overline{\phi'(x)} i_\gamma(d^4x) \psi'(x) = i \int_{\Sigma_t} \overline{\phi'(x)} \overleftarrow{D^A} v_t(x) \not{n}_t(x) i_\gamma(d^4x) \psi'(x). \tag{133}$$

We consider the function

$$F : T \times T \rightarrow \mathbb{C}, \quad F(s, t) := \int_{x \in \Sigma_s} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4x) k(x, y) i_\gamma(d^4y) \psi(y). \tag{134}$$

We apply (132) to $\phi' = \phi$ and $\psi'(x) = \int_{y \in \Sigma_t} k(x, y) i_\gamma(d^4y) \psi(y)$ to get

$$\frac{\partial}{\partial s} F(s, t) = -i \int_{x \in \Sigma_s} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4x) v_s(x) \not{n}_s(x) D_x^A k(x, y) i_\gamma(d^4y) \psi(y). \tag{135}$$

Similarly, we apply (133) to $\overline{\phi'(y)} = \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4x) k(x, y)$ and $\psi' = \psi$ to get

$$\frac{\partial}{\partial t} F(s, t) = i \int_{x \in \Sigma_s} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4x) k(x, y) \overleftarrow{D_y^A} v_t(y) \not{n}_t(y) i_\gamma(d^4y) \psi(y). \tag{136}$$

From the chain rule, claim (125) follows:

$$\begin{aligned}
& \frac{d}{dt} F(t, t) \\
&= -i \int_{x \in \Sigma_s} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4x) [v_t(x) \not{n}_t(x) D_x^A k(x, y) - k(x, y) \overleftarrow{D_y^A} v_t(y) \not{n}_t(y)] i_\gamma(d^4y) \psi(y).
\end{aligned} \tag{137}$$

□

From formula (125) and the chain rule, we immediately get the following corollary.

Corollary 2.12. *For any smooth function $k : \mathbb{R}^4 \times \mathbb{R}^4 \times T \rightarrow \mathbb{C}^{4 \times 4}$, $(x, y, t) \mapsto k_t(x, y)$, any $\phi, \psi \in \mathcal{C}_A$, and any $t \in T$ we have*

$$\begin{aligned} & \frac{d}{dt} \int_{x \in \Sigma_t} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4 x) k_t(x, y) i_\gamma(d^4 y) \psi(y) \\ &= \int_{x \in \Sigma_t} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4 x) \left[-i \mathcal{D}_t^A k(x, y) + \frac{\partial k_t}{\partial t}(x, y) \right] i_\gamma(d^4 y) \psi(y). \end{aligned} \quad (138)$$

This completes step one, and next, we turn to the relevant estimates. In the following calculations for fixed $t \in T$, we drop the index t in $v = v_t$ and $n = n_t$. Also, the t -dependence of the remainder terms r_{\dots} is suppressed in the notation below, as we have uniformity in t of the error bounds. Recall from equation (42) that $E_\mu = F_{\mu\nu} n^\nu$ denotes the “electric field” of the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with respect to the local Cauchy surface Σ .

Lemma 2.13. *For $u \in \text{Past}$, $\epsilon > 0$, and $x, y \in \mathbb{R}^4$, let*

$$p^{A, \epsilon u}(x, y) := e^{-i\lambda^A(x, y)} p^-(y - x + i\epsilon u) \quad (139)$$

with λ^A defined in (38). Then for $t \in T$, $x, y \in \Sigma_t$, $z = (z^0, \mathbf{z}) = y - x$, and $w = z + i\epsilon u$ we have

$$\begin{aligned} & \mathcal{D}_t^A p^{A, \epsilon u}(x, y) \\ &= \frac{1}{2} v(x) \not{n}(x) \gamma^\nu F_{\mu\nu}(x) z^\mu p^{A, \epsilon u}(x, y) + \frac{1}{2} p^{A, \epsilon u}(x, y) \gamma^\nu F_{\mu\nu}(y) z^\mu v(y) \not{n}(y) + r_2(x, y, \epsilon u) \end{aligned} \quad (140)$$

$$= -\frac{i}{2m} v(x) z^\mu E_\mu(x) \not{D}(w) + r_3(x, y, \epsilon u) + r_4(x, y, \epsilon u) \quad (141)$$

with error terms

$$r_2 = O_{A, u, \Sigma} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|} \right) [1_K(x) \vee 1_K(y)], \quad (142)$$

$$r_3 = O_{A, u, \Sigma} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|} \right) [1_K(x) \vee 1_K(y)], \quad (143)$$

$$r_4 = O_{A, u, \Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) [1_K(x) \vee 1_K(y)] \quad (144)$$

for any compact set K containing the support of A . For any two different points $x \neq y$ in Σ_t , the limit $r_3(x, y, 0) := \lim_{\epsilon \downarrow 0} r_3(x, y, \epsilon u)$ exists.

Proof. We calculate for $x, y \in \Sigma_t$, $u \in \text{Past}$, and $\epsilon > 0$:

$$\begin{aligned} & D_x^A [e^{-i\lambda^A(x, y)} p^-(y - x + i\epsilon u)] \\ &= [\not{\partial}^x \lambda^A(x, y) - \not{A}(x)] e^{-i\lambda^A(x, y)} p^-(y - x + i\epsilon u) + e^{-i\lambda^A(x, y)} (i\not{\partial}^x - m) p^-(y - x + i\epsilon u) \\ &= [\not{\partial}^x \lambda^A(x, y) - \not{A}(x)] p^{A, \epsilon u}(x, y), \quad \text{because} \quad (i\not{\partial}^x - m) p^-(y - x + i\epsilon u) = 0. \end{aligned} \quad (145)$$

Using the definition (38) of λ^A , we get

$$\begin{aligned}\not{\partial}^x \lambda^A(x, y) - \not{A}(x) &= \frac{1}{2} \gamma^\nu [A_\nu(y) - A_\nu(x) + (x^\mu - y^\mu) \partial_\nu^x A_\mu(x)] \\ &= \frac{1}{2} [\gamma^\nu F_{\mu\nu}(x)(y^\mu - x^\mu) + r_5(x, y)]\end{aligned}\quad (146)$$

with the Taylor rest term

$$\begin{aligned}r_5(x, y) &= \gamma^\nu [A_\nu(y) - A_\nu(x) - (y^\mu - x^\mu) \partial_\mu^x A_\nu(x)] = O_A(|x - y|^2) [1_K(x) \vee 1_K(y)] \\ &= O_A(|\mathbf{z}|^2) [1_K(x) \vee 1_K(y)] \text{ with } \mathbf{z} = \mathbf{y} - \mathbf{x};\end{aligned}\quad (147)$$

cf. formula (28) in the appendix, which compares $|z|$ with $|\mathbf{z}|$. Recall that K denotes a compact set containing the support of A . Similarly, we find

$$\begin{aligned}[e^{-i\lambda^A(x, y)} p^-(y - x + i\epsilon u)] \overleftarrow{D}_y^A \\ = e^{-i\lambda^A(x, y)} p^-(y - x + i\epsilon u) [-\not{\partial}^y \lambda^A(x, y) - \not{A}(y)] + p^-(y - x + i\epsilon u) (-i\overleftarrow{\not{\partial}}^y - m) e^{-i\lambda^A(x, y)} \\ = p^{A, \epsilon u}(x, y) [-\not{\partial}^y \lambda^A(x, y) - \not{A}(y)].\end{aligned}\quad (148)$$

Using the symmetry $\lambda^A(x, y) = -\lambda^A(y, x)$ and interchanging x and y , equation (146) can be rewritten in the form

$$-\not{\partial}^y \lambda^A(x, y) - \not{A}(y) = \frac{1}{2} [-\gamma^\nu F_{\mu\nu}(y)(y^\mu - x^\mu) + r_5(y, x)].\quad (149)$$

Combining this with the definition (126) of \mathcal{D}_t^A , we find for $x, y \in \Sigma_t$, $z = y - x$

$$\begin{aligned}\mathcal{D}_t^A p^{A, \epsilon u}(x, y) \\ = \frac{1}{2} v(x) \not{n}(x) [\gamma^\nu F_{\mu\nu}(x) z^\mu + r_5(x, y)] p^{A, \epsilon u}(x, y) \\ + \frac{1}{2} p^{A, \epsilon u}(x, y) [\gamma^\nu F_{\mu\nu}(y) z^\mu - r_5(y, x)] v(y) \not{n}(y) \\ = \frac{1}{2} v(x) \not{n}(x) \gamma^\nu F_{\mu\nu}(x) z^\mu p^{A, \epsilon u}(x, y) + \frac{1}{2} p^{A, \epsilon u}(x, y) \gamma^\nu F_{\mu\nu}(y) z^\mu v(y) \not{n}(y) + r_2(x, y, \epsilon u)\end{aligned}\quad (150)$$

with the error term

$$\begin{aligned}r_2(x, y, \epsilon u) &= \frac{1}{2} v(x) \not{n}(x) r_5(x, y) p^{A, \epsilon u}(x, y) - \frac{1}{2} p^{A, \epsilon u}(x, y) r_5(y, x) v(y) \not{n}(y) \\ &= O_{A, u, \Sigma} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|} \right) [1_K(x) \vee 1_K(y)],\end{aligned}\quad (151)$$

for $t \in T$, $x, y \in \Sigma_t$, $\epsilon > 0$, $u \in \text{Past}$. Here we used the bound (238) in Lemma A.1 in the appendix for p^- , the quadratic bound (147) for $r_5(x, y)$, and the fact that $|vn|$, being continuous, is bounded on compact sets. This proves the claim given in (140) with the error bound (142).

It remains to prove the claim given in (141) with the bounds (143) and (144). Recall the definitions of $p^{A,\epsilon u}$ and p^- given in (139) and (30), respectively. We have

$$p^{A,\epsilon u}(x, y) = -\frac{i}{2m}\not\partial D(w) + r_6(x, y, \epsilon u) \quad (152)$$

with the error term

$$r_6(x, y, \epsilon u) = \frac{1}{2}e^{-i\lambda^A(x,y)}D(w) + (e^{-i\lambda^A(x,y)} - 1)p^-(z + i\epsilon u) = O_{A,u,\Sigma}\left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^2}\right) \quad (153)$$

using the bounds (232), (238) from the appendix and the Taylor bound

$$|e^{-i\lambda^A(x,y)} - 1| = O_A(|z|) \leq O_{A,\Sigma}(|\mathbf{z}|), \quad (154)$$

which follows from $\lambda^A \in \mathcal{G}(A)$, cf. Definition 2.2 and, once more, from the estimate (28) in the appendix. Hence we get from (150)

$$\begin{aligned} \mathcal{D}_t^A p^{A,\epsilon u}(x, y) - r_2(x, y, \epsilon u) &= \frac{1}{2}v(x)\not\partial(x)\gamma^\nu F_{\mu\nu}(x)z^\mu p^{A,\epsilon u}(x, y) + \frac{1}{2}p^{A,\epsilon u}(x, y)\gamma^\nu F_{\mu\nu}(y)z^\mu v(y)\not\partial(y) \\ &= -\frac{i}{4m}v(x)\not\partial(x)\gamma^\nu F_{\mu\nu}(x)z^\mu \not\partial D(w) - \frac{i}{4m}\not\partial D(w)\gamma^\nu F_{\mu\nu}(y)z^\mu v(y)\not\partial(y) + r_7(x, y, \epsilon u) \end{aligned} \quad (155)$$

with the error term

$$r_7 = \frac{1}{2}v(x)\not\partial(x)\gamma^\nu F_{\mu\nu}(x)z^\mu r_6 + \frac{1}{2}r_6\gamma^\nu F_{\mu\nu}(y)z^\mu v(y)\not\partial(y) = O_{A,u,\Sigma}\left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|}\right)[1_K(x) \vee 1_K(y)]. \quad (156)$$

We employ estimate (236) for ∂D from the appendix and the fact $\text{supp } F_{\mu\nu} \subseteq K$ to find

$$v(x)\not\partial(x)F_{\mu\nu}(x)\gamma^\nu z^\mu \not\partial D(w) = v(x)\not\partial(x)F_{\mu\nu}(x)\gamma^\nu w^\mu \not\partial D(w) + r_8(x, y, \epsilon u) \quad (157)$$

$$\not\partial D(w)\gamma^\nu F_{\mu\nu}(y)z^\mu v(y)\not\partial(y) = \not\partial D(w)\gamma^\nu F_{\mu\nu}(y)w^\mu v(y)\not\partial(y) + r_9(x, y, \epsilon u) \quad (158)$$

with the error terms

$$r_8 = -v(x)\not\partial(x)F_{\mu\nu}(x)\gamma^\nu i\epsilon u^\mu \not\partial D(w) = O_{A,u,\Sigma}\left(\sqrt{\epsilon}\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}}\right)1_K(x), \quad (159)$$

$$r_9 = -\not\partial D(w)\gamma^\nu F_{\mu\nu}(y)i\epsilon u^\mu v(y)\not\partial(y) = O_{A,u,\Sigma}\left(\sqrt{\epsilon}\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}}\right)1_K(y). \quad (160)$$

Substituting this in (155), we conclude

$$\begin{aligned} \mathcal{D}_t^A p^{A,\epsilon u}(x, y) &= -\frac{i}{4m}v(x)\not\partial(x)\gamma^\nu F_{\mu\nu}(x)w^\mu \not\partial D(w) - \frac{i}{4m}\not\partial D(w)\gamma^\nu F_{\mu\nu}(y)w^\mu v(y)\not\partial(y) \\ &\quad + (r_2 + r_7 + r_{10})(x, y, \epsilon u) \end{aligned} \quad (161)$$

with the additional error term

$$r_{10} = -\frac{i}{4m}(r_8 + r_9) = O_{A,u,\Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) [1_K(x) \vee 1_K(y)]. \quad (162)$$

The following ‘‘Lorentz symmetry relation’’ will be used several times in the calculations below.

$$w_\nu \partial_\mu D(w) = w_\mu \partial_\nu D(w) \quad \text{for} \quad w \in \text{domain}(r). \quad (163)$$

Equation (163) can be seen as follows. Using $D = f \circ r$ with $f(\xi) = -m^3(2\pi^2)^{-1}K_1(m\xi)/(m\xi)$ from (31) and $\partial_\mu r(w) = -\frac{w_\mu}{r(w)}$, we obtain $w_\nu \partial_\mu D(w) = -\frac{w_\nu w_\mu}{r(w)} f'(r(w)) = w_\mu \partial_\nu D(w)$.

Using the anticommutator relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ for the Dirac-matrices three times and the Lorentz symmetry relation (163), we calculate

$$\begin{aligned} v(x) \not{n}(x) F_{\mu\nu}(x) \gamma^\nu w^\mu \not{\partial} D(w) &= [\not{n}(x) \gamma^\nu \psi] v(x) F_{\mu\nu}(x) \partial^\mu D(w) \\ &= [2n^\nu(x) \psi - 2\gamma^\nu n_\sigma(x) w^\sigma + 2w^\nu \not{n}(x) - \psi \gamma^\nu \not{n}(x)] v(x) F_{\mu\nu}(x) \partial^\mu D(w) \\ &= 2n^\nu(x) \psi v(x) F_{\mu\nu}(x) \partial^\mu D(w) \end{aligned} \quad (164)$$

$$- 2\gamma^\nu n_\sigma(x) w^\sigma v(x) F_{\mu\nu}(x) \partial^\mu D(w) \quad (165)$$

$$+ 2w^\nu \not{n}(x) v(x) F_{\mu\nu}(x) \partial^\mu D(w) \quad (166)$$

$$- \psi \gamma^\nu \not{n}(x) v(x) F_{\mu\nu}(x) \partial^\mu D(w). \quad (167)$$

For the first term (164), using the Lorentz symmetry (163) again, we get

$$\begin{aligned} (164) &= 2n^\nu(x) \psi v(x) F_{\mu\nu}(x) \partial^\mu D(w) = 2v(x) w^\mu E_\mu(x) \not{\partial} D(w) \\ &= 2v(x) z^\mu E_\mu(x) \not{\partial} D(w) + r_{11}(x, y, \epsilon u) \end{aligned} \quad (168)$$

with the error term

$$r_{11} = 2v(x) i\epsilon u^\mu E_\mu(x) \not{\partial} D(w) = O_{A,u,\Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) 1_K(x), \quad (169)$$

where in the last step we have used estimate (236) once more. For the second term (165), we use $n_\sigma(x) z^\sigma = O_\Sigma(|\mathbf{z}|^2)$, which holds because of $x, y \in \Sigma_t$ and $n(x) \perp T_x \Sigma_t$, to get

$$(165) = -2\gamma^\nu n_\sigma(x) w^\sigma v(x) F_{\mu\nu}(x) \partial^\mu D(w) = r_{12}(x, y, \epsilon u) + r_{13}(x, y, \epsilon u) \quad (170)$$

with the error terms

$$r_{12} = -2\gamma^\nu n_\sigma(x) z^\sigma v(x) F_{\mu\nu}(x) \partial^\mu D(w) = O_{A,u,\Sigma} \left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|} \right) 1_K(x), \quad (171)$$

$$r_{13} = -2\gamma^\nu n_\sigma(x) i\epsilon u^\sigma v(x) F_{\mu\nu}(x) \partial^\mu D(w) = O_{A,u,\Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) 1_K(x). \quad (172)$$

We have used the estimates (234) and, once more, (236). The contribution of the third term (166) is zero, i.e.

$$(166) = 2w^\nu \not{n}(x) v(x) F_{\mu\nu}(x) \partial^\mu D(w) = 0, \quad (173)$$

because of symmetry $w^\nu \partial^\mu D(w) = w^\mu \partial^\nu D(w)$, cf. (163), and antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$. To express the fourth term (167), we use the Lorentz symmetry relation (163) again and replace x by y up to the following error term:

$$r_{14}(x, y) = F_{\mu\nu}(x)v(x)\not{n}(x) - F_{\mu\nu}(y)v(y)\not{n}(y) = O_{A,\Sigma}(|\mathbf{z}|)[1_K(x) \vee 1_K(y)]. \quad (174)$$

We obtain for the fourth term (167):

$$\begin{aligned} (167) &= -\psi \partial^\mu D(w) \gamma^\nu \not{n}(x) v(x) F_{\mu\nu}(x) = -w^\mu \not{\partial} D(w) \gamma^\nu F_{\mu\nu}(x) v(x) \not{n}(x) \\ &= -\not{\partial} D(w) \gamma^\nu F_{\mu\nu}(y) w^\mu v(y) \not{n}(y) + r_{15}(x, y, \epsilon u) \end{aligned} \quad (175)$$

with the error term

$$r_{15} = w^\mu \not{\partial} D(w) \gamma^\nu r_{14} = O_{A,u,\Sigma} \left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|} \right) [1_K(x) \vee 1_K(y)]. \quad (176)$$

We have used estimate (235) from the appendix and the bound (174). The expressions (168), (170), (173) and (175) of the four terms (164)-(167) give

$$\begin{aligned} v(x)\not{n}(x)F_{\mu\nu}(x)\gamma^\nu w^\mu \not{\partial} D(w) &= (164) + (165) + (166) + (167) \\ &= [2v(x)z^\mu E_\mu(x)\not{\partial} D(w) + r_{11}] + [r_{12} + r_{13}] + 0 + [-\not{\partial} D(w)\gamma^\nu F_{\mu\nu}(y)w^\mu v(y)\not{n}(y) + r_{15}], \end{aligned} \quad (177)$$

which can be rewritten in the form

$$\begin{aligned} v(x)\not{n}(x)F_{\mu\nu}(x)\gamma^\nu w^\mu \not{\partial} D(w) + \not{\partial} D(w)\gamma^\nu F_{\mu\nu}(y)w^\mu v(y)\not{n}(y) \\ = 2v(x)z^\mu E_\mu(x)\not{\partial} D(w) + r_{16}(x, y, \epsilon u) + r_{17}(x, y, \epsilon u) \end{aligned} \quad (178)$$

with the error terms

$$r_{16} = r_{12} + r_{15} = O_{A,u,\Sigma} \left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|} \right) [1_K(x) \vee 1_K(y)], \quad (179)$$

$$r_{17} = r_{11} + r_{13} = O_{A,u,\Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) [1_K(x) \vee 1_K(y)]. \quad (180)$$

We have used the estimates (171) and (176) to bound r_{16} and the estimates (169) and (172) to bound r_{17} . Substituting this result in equation (161) together with the error bounds (151), (156) and (162), we infer

$$\begin{aligned} \mathcal{D}_t^A p^{A,\epsilon u}(x, y) \\ = -\frac{i}{4m} v(x)\not{n}(x)F_{\mu\nu}(x)\gamma^\nu w^\mu \not{\partial} D(w) - \frac{i}{4m} \not{\partial} D(w)\gamma^\nu F_{\mu\nu}(y)w^\mu v(y)\not{n}(y) + r_2 + r_7 + r_{10} \\ = -\frac{i}{2m} v(x)z^\mu E_\mu(x)\not{\partial} D(w) + r_3 + r_4 \end{aligned} \quad (181)$$

with the error terms

$$r_3(x, y, \epsilon u) = r_2 + r_7 - \frac{i}{4m} r_{16} = O_{A,u,\Sigma} \left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|} \right) [1_K(x) \vee 1_K(y)], \quad (182)$$

$$r_4(x, y, \epsilon u) = r_{10} - \frac{i}{4m} r_{17} = O_{A,u,\Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) [1_K(x) \vee 1_K(y)]. \quad (183)$$

This proves the claim given in (141) with the bounds (143), (144). Recall that despite the uniformity in ϵ of the bound given in (182), $r_3 = r_3(x, y, \epsilon u)$ depends on ϵ . To ensure existence of the limit $\lim_{\epsilon \downarrow 0} r_3(x, y, \epsilon u)$ for two different points $x, y \in \Sigma_t$ from the explicit form of r_3 , we observe that $z = y - x$ is space-like, and hence $z \in \text{domain } r$. As a consequence, the functions D and $\partial_\mu D$ are continuous at z , cf. Lemma 2.1, which implies the claim. \square

In the following, we abbreviate $\partial_\mu = \partial/\partial w^\mu$. Recall the notation $r(w) = \sqrt{-w_\mu w^\mu}$ from (27).

Lemma 2.14. *For $w \in \text{domain}(r)$ and $\mu = 0, 1, 2, 3$, one has*

$$\partial_\mu[r(w)^2 \not\partial D(w)] = 2w_\mu \not\partial D(w) - \gamma_\mu w^\nu \partial_\nu D(w) + \psi w_\mu m^2 D(w). \quad (184)$$

Proof. The function D fulfills the Klein-Gordon equation

$$(\square + m^2)D(w) = 0, \quad w \in \text{domain}(r). \quad (185)$$

Indeed, for $w \in \mathbb{R}^4 + i\text{Past}$, this can be seen from the definition (31) of D as follows: Because of the fast convergence of e^{ipw} to 0 as $|p| \rightarrow \infty$, $p \in \mathcal{M}_-$, we can interchange the Klein-Gordon-operator with the integral in the following calculation:

$$\begin{aligned} (\square + m^2)D(w) &= (2\pi)^{-3} m^{-1} \int_{\mathcal{M}_-} (\square^w + m^2) e^{ipw} i_p(d^4 p) \\ &= (2\pi)^{-3} m^{-1} \int_{\mathcal{M}_-} (-p^2 + m^2) e^{ipw} i_p(d^4 p) = 0. \end{aligned} \quad (186)$$

By analytic continuation, the Klein-Gordon equation (185) follows for all $w \in \text{domain}(r)$. Equation (184) is proven by the following calculation:

$$\begin{aligned} \partial_\mu[r(w)^2 \not\partial D(w)] &= -\partial_\mu[w^\nu w_\nu \not\partial D(w)] \\ &\stackrel{(163)}{=} -\partial_\mu[w^\nu \psi \partial_\nu D(w)] \\ &= -\psi \partial_\mu D(w) - w^\nu \gamma_\mu \partial_\nu D(w) - w^\nu \psi \partial_\mu \partial_\nu D(w) \\ &= -\psi \partial_\mu D(w) - w^\nu \gamma_\mu \partial_\nu D(w) - \psi \partial_\nu (w^\nu \partial_\mu D(w)) + \psi (\partial_\nu w^\nu) \partial_\mu D(w) \\ &= 3\psi \partial_\mu D(w) - w^\nu \gamma_\mu \partial_\nu D(w) - \psi \partial_\nu (w^\nu \partial_\mu D(w)) \\ &\stackrel{(163)}{=} 3\psi \partial_\mu D(w) - w^\nu \gamma_\mu \partial_\nu D(w) - \psi \partial_\nu (w_\mu \partial^\nu D(w)) \\ &= 2\psi \partial_\mu D(w) - w^\nu \gamma_\mu \partial_\nu D(w) - \psi w_\mu \square D(w) \\ &\stackrel{(163), (185)}{=} 2w_\mu \not\partial D(w) - \gamma_\mu w^\nu \partial_\nu D(w) + \psi w_\mu m^2 D(w). \end{aligned} \quad (187)$$

\square

Recall the definition of the helper object $s_\Sigma^{A, \epsilon u}(x, y) = [(\not\partial E)(x)][(r^2 \not\partial D)(w)]/(8m)$ introduced in Definition 2.6. The properties of $s_\Sigma^{A, \epsilon u}(x, y)$ claimed in Lemma 2.7 follow analogously to the arguments used in (92)–(95), i.e., from the bound (233) given in Corollary A.1 in the appendix, the compact support of E , boundedness of $\partial(\not\partial_t E_t)/\partial t$, and the dominated convergence theorem.

Lemma 2.15. For $t \in \mathbb{R}$, $x, y \in \Sigma_t$, $z = y - x$, $u \in \text{Past}$, and $\epsilon > 0$ we have

$$\mathcal{D}_t^A s_{\Sigma}^{A, \epsilon u}(x, y) = \frac{i}{2m} v_t(x) z^\mu E_\mu(x) \not\partial D(w) + r_{18}(x, y, \epsilon u) + r_{19}(x, y, \epsilon u), \quad (188)$$

$$\mathcal{D}_t^A (p_{\Sigma}^{A, \epsilon u} + s_{\Sigma}^{A, \epsilon u})(x, y) = r_{20}(x, y, \epsilon u) + r_{21}(x, y, \epsilon u) \quad (189)$$

with error terms that fulfill the bounds

$$\begin{aligned} r_{18} &= O_{A, u, \Sigma} \left(\frac{e^{-C_{12}|z|}}{|z|} \right) 1_K(x), & r_{19} &= O_{A, u, \Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_{12}|z|}}{|z|^{5/2}} \right) 1_K(x), \\ r_{20} &= O_{A, u, \Sigma} \left(\frac{e^{-C_{12}|z|}}{|z|} \right) [1_K(x) \vee 1_K(y)], & r_{21} &= O_{A, u, \Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_{12}|z|}}{|z|^{5/2}} \right) [1_K(x) \vee 1_K(y)] \end{aligned} \quad (190)$$

$$(191)$$

with some positive constant $C_{12}(\Sigma)$. Furthermore, for $x \neq y$ the following limit exists:

$$r_{20}(x, y, 0) := \lim_{\epsilon \downarrow 0} r_{20}(x, y, \epsilon u) \quad (192)$$

Proof. In this proof, we abbreviate $w = y - x + i\epsilon u = z + iu\epsilon$. Moreover, we suppress the w dependence of $r(w)$, $D(w)$, ∂^w and again also the t -dependence of v , n , and of the remainder terms r_{\dots} in the notation. Using the definition of \mathcal{D}_t^A given in (126) of Lemma 2.11, we get

$$\begin{aligned} &8m \mathcal{D}_t^A s_{\Sigma_t}^{A, \epsilon u}(x, y) \\ &= v(x) \not{n}(x) D_x^A [\not{n}(x) \not{E}(x) r^2 \not\partial D] - [\not{n}(x) \not{E}(x) r^2 \not\partial D] \overleftarrow{D}_y^A \not{n}(y) v(y) \\ &= v(x) \not{n}(x) i \not{\partial}^x [\not{n}(x) \not{E}(x) r^2 \not\partial D] - [\not{n}(x) \not{E}(x) r^2 \not\partial D] \overleftarrow{\not{\partial}}^y (-i) \not{n}(y) v(y) + r_{22}(x, y, \epsilon u) \\ &= i v(x) \not{n}(x) \gamma^\mu \not{n}(x) \not{E}(x) \partial_\mu^x [r^2 \not\partial D] + i \not{n}(x) \not{E}(x) \partial_\mu^y [r^2 \not\partial D] \gamma^\mu \not{n}(y) v(y) + r_{23}(x, y, \epsilon u) \\ &= -i v(x) \not{n}(x) \gamma^\mu \not{n}(x) \not{E}(x) \partial_\mu [r^2 \not\partial D] + i \not{n}(x) \not{E}(x) \partial_\mu [r^2 \not\partial D] \gamma^\mu \not{n}(x) v(x) + r_{24}(x, y, \epsilon u), \end{aligned} \quad (193)$$

where the remainder terms are defined and estimated as follows:

- (i) Recalling the definitions (123) and (124) of the Dirac operators D_A and \overleftarrow{D}^A and the fact that A is compactly supported, the estimate (233) of Corollary A.1 in the appendix ensures

$$\begin{aligned} r_{22} &= v(x) \not{n}(x) (-m - A(x)) [\not{n}(x) \not{E}(x) r^2 \not\partial D] - [\not{n}(x) \not{E}(x) r^2 \not\partial D] (-m - A(y)) \not{n}(y) v(y) \\ &= O_{A, u, \Sigma} \left(\frac{e^{-C_D|z|}}{|z|} \right) 1_K(x) \end{aligned} \quad (194)$$

for some compact set K containing the support of E .

- (ii) Using once more that E has compact support and using the bound (233) again we have the analogous estimate

$$\begin{aligned} r_{23} &= r_{22} + i v(x) \not{n}(x) \gamma^\mu (\partial_\mu^x [\not{n}(x) \not{E}(x)]) r^2 \not\partial D + i (\partial_\mu^y [\not{n}(x) \not{E}(x)]) r^2 \not\partial D \gamma^\mu \not{n}(y) v(y) \\ &= O_{A, u, \Sigma} \left(\frac{e^{-C_D|z|}}{|z|} \right) 1_K(x) \end{aligned} \quad (195)$$

(iii) Using the signs coming from inner derivatives: $-\partial^x D(w) = \partial^y D(w) = \partial D(w)$ and the Taylor expansion

$$\not{n}(y)v(y) = \not{n}(x)v(x) + r_{25}(x, y) \quad \text{with} \quad r_{25} = O_{\Sigma}(|\mathbf{z}|) \quad (196)$$

for $x, y \in \Sigma_t$ with $x \in K$ we find with the help of bound (237) in the appendix:

$$r_{24} = r_{23} + i\not{n}(x)\not{E}(x)\partial_{\mu}[r^2\check{\phi}D]\gamma^{\mu}r_{25} = O_{A,u,\Sigma}\left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|}\right)1_K(x). \quad (197)$$

In the following calculations, we drop the argument x ; thus, v , n , and E stand for $v(x)$, $n(x)$, and $E(x)$, respectively, but $r = r(w)$ and $D = D(w)$. Using Lemma 2.14, we get

$$\begin{aligned} -i\left(8m\mathcal{D}_t^A s_{\Sigma}^{A,\epsilon u}(x, y) - r_{24}\right) &= -v\not{n}\gamma^{\mu}\not{n}\not{E}\partial_{\mu}[r^2\check{\phi}D] + \not{n}\not{E}\partial_{\mu}[r^2\check{\phi}D]\gamma^{\mu}\not{n}v \\ &= -v\not{n}\gamma^{\mu}\not{n}\not{E}[2w_{\mu}\check{\phi}D - \gamma_{\mu}w^{\nu}\partial_{\nu}D] + v\not{n}\not{E}[2w_{\mu}\check{\phi}D - \gamma_{\mu}w^{\nu}\partial_{\nu}D]\gamma^{\mu}\not{n} + r_{26}(x, y, \epsilon u) \\ &= T_1 + T_2 + T_3 + T_4 + r_{26} \end{aligned} \quad (198)$$

with the four terms

$$\begin{aligned} T_1 &= -2v\not{n}\psi\not{n}\not{E}\check{\phi}D, & T_2 &= v\not{n}\gamma^{\mu}\not{n}\not{E}\gamma_{\mu}w^{\nu}\partial_{\nu}D, \\ T_3 &= 2v\not{n}\not{E}\check{\phi}D\psi\not{n}, & T_4 &= -v\not{n}\not{E}\gamma_{\mu}\gamma^{\mu}\not{n}w^{\nu}\partial_{\nu}D, \end{aligned} \quad (199)$$

and the remainder term

$$r_{26} = -v\not{n}\gamma^{\mu}\not{n}\not{E}\psi w_{\mu}m^2D + v\not{n}\not{E}\psi w_{\mu}m^2D\gamma^{\mu}\not{n} = O_{A,u,\Sigma}(e^{-C_D|\mathbf{z}|})1_K(x), \quad (200)$$

where the bound comes from (231) of Corollary A.1 in the appendix and from $\text{supp } E \subseteq K$. We evaluate the four terms T_j separately. Using the anticommutation rules $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$ for the Dirac matrices and $\not{n}^2 = 1$, we get

$$\begin{aligned} T_1 &= -2v\not{n}[2w^{\nu}n_{\nu} - \not{n}\psi]\not{E}\check{\phi}D \\ &= -4v\not{n}w^{\nu}n_{\nu}\not{E}\check{\phi}D + 2v[2w^{\mu}E_{\mu} - \not{E}\psi]\check{\phi}D \\ &= -4v\not{n}w^{\nu}n_{\nu}\not{E}\check{\phi}D + 4vw^{\mu}E_{\mu}\check{\phi}D - 2v\not{E}w^{\mu}\partial_{\mu}D, \end{aligned} \quad (201)$$

where in the last step we used the Lorentz symmetry (163) to compute

$$\begin{aligned} \psi\check{\phi}D &= \gamma^{\mu}\gamma^{\nu}w_{\mu}\partial_{\nu}D = \frac{1}{2}(\gamma^{\mu}\gamma^{\nu}w_{\mu}\partial_{\nu}D + \gamma^{\mu}\gamma^{\nu}w_{\nu}\partial_{\mu}D) \\ &= \frac{1}{2}(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu})w_{\mu}\partial_{\nu}D = w^{\mu}\partial_{\mu}D. \end{aligned} \quad (202)$$

Using the anticommutation rules again, the fact $\gamma^{\mu}\gamma_{\mu} = 4$, the definition $E_{\mu} = F_{\mu\nu}n^{\nu}$ given in (42), and the antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$, we get

$$\begin{aligned} \gamma^{\mu}\not{n}\not{E}\gamma_{\mu} &= (2n^{\mu} - \not{n}\gamma^{\mu})(2E_{\mu} - \gamma_{\mu}\not{E}) = 4n^{\mu}E_{\mu} - 4\not{n}\not{E} + \not{n}\gamma^{\mu}\gamma_{\mu}\not{E} \\ &= 4n^{\mu}E_{\mu} = 4n^{\mu}F_{\mu\nu}n^{\nu} = 0 \end{aligned} \quad (203)$$

and therefore $T_2 = 0$. Using the same argument that was used to derive (202) we also find $\not\partial D(w)\psi = w^\mu \partial_\mu D$, and hence,

$$T_3 = 2v\not{n}\not{E}w^\mu \partial_\mu D\not{n}. \quad (204)$$

Finally, we have

$$T_4 = -4v\not{n}\not{E}\not{n}w^\nu \partial_\nu D, \quad (205)$$

which yields

$$T_3 + T_4 = -2v\not{n}\not{E}\not{n}w^\mu \partial_\mu D = 2v\not{n}^2 \not{E}w^\mu \partial_\mu D = 2v\not{E}w^\mu \partial_\mu D. \quad (206)$$

We have used that \not{n} and \not{E} anticommute because of $n^\mu E_\mu = n^\mu F_{\mu\nu} n^\nu = 0$. Together with the expression (201) for T_1 and $T_2 = 0$, we conclude

$$T_1 + T_2 + T_3 + T_4 = 4vw^\mu E_\mu \not\partial D + r_{27}(x, y, \epsilon u). \quad (207)$$

with the error terms

$$r_{27} = -4v\not{n}w^\nu n_\nu \not{E}\not\partial D = r_{28}(x, y, \epsilon u) + r_{29}(x, y, \epsilon u), \quad (208)$$

where using $w = z + i\epsilon u$

$$r_{28} = -4v\not{n}i\epsilon u^\nu n_\nu \not{E}\not\partial D, \quad r_{29} = -4v\not{n}z^\nu n_\nu \not{E}\not\partial D. \quad (209)$$

Inequality (236) from the appendix and the fact $\text{supp } E \subseteq K$ provide the bound

$$r_{28} = O_{A,u,\Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) 1_K(x). \quad (210)$$

For the next estimate, we observe $(\nabla t_{\Sigma_t}(\mathbf{x}) \cdot \mathbf{z}, \mathbf{z}) \in T_x \Sigma_t \perp n(x)$; recall the parametrization (13) of Σ_t . We obtain the Taylor expansion

$$z^\nu n_\nu = n_0(x)[t_{\Sigma_t}(\mathbf{y}) - t_{\Sigma_t}(\mathbf{x})] - \mathbf{n}(x) \cdot \mathbf{z} = n_0(x)\nabla t_{\Sigma_t}(\mathbf{x}) \cdot \mathbf{z} - \mathbf{n}(x) \cdot \mathbf{z} + O_\Sigma(|\mathbf{z}|^2) = O_\Sigma(|\mathbf{z}|^2) \quad (211)$$

uniformly for x in the compact set K . Using (234) from the appendix and the support property of E again, this implies

$$r_{29} = O_{A,u,\Sigma} \left(\frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|} \right) 1_K(x). \quad (212)$$

Finally, we have from equation (207)

$$T_1 + T_2 + T_3 + T_4 - r_{27} = 4vw^\mu E_\mu \not\partial D = 4vz^\mu E_\mu \not\partial D + r_{30}(x, y, \epsilon u) \quad (213)$$

with the error term

$$r_{30} = 4vi\epsilon u^\mu E_\mu \not\partial D = O_{A,u,\Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D|\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) 1_K(x), \quad (214)$$

where once again we have used the bound (236) from the appendix and the fact $\text{supp } E \subseteq K$. Let us summarize: We use the equations (198), (213), and (208) to get the claimed formula

$$\mathcal{D}_t^A s_{\Sigma_t}^{A, \epsilon u}(x, y) = \frac{i}{2m} v z^\mu E_\mu \not{D} + r_{18} + r_{19} \quad (188)$$

with the remainder terms

$$r_{18} := \frac{r_{24}}{8m} + \frac{i}{8m} (r_{26} + r_{29}) = O_{A, u, \Sigma} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|} + e^{-C_D |\mathbf{z}|} \right) 1_K(x) = O_{A, u, \Sigma} \left(\frac{e^{-C_{12} |\mathbf{z}|}}{|\mathbf{z}|} \right) 1_K(x) \quad (215)$$

$$r_{19} := \frac{i}{8m} (r_{28} + r_{30}) = O_{A, u, \Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) 1_K(x) = O_{A, u, \Sigma} \left(\sqrt{\epsilon} \frac{e^{-C_{12} |\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right) 1_K(x) \quad (216)$$

with any positive constant $C_{12}(\Sigma) < C_D(\Sigma)$. We have applied the error bounds (197), (200), and (212) for the first remainder term r_{18} , and the bounds (210) and (214) for the second remainder term r_{19} . Finally, we have weakened the bounds slightly to get a simpler notation. This shows the claimed error bounds in (190).

Combining this with Lemma 2.13 and setting $r_{20} = r_3 + r_{18}$, $r_{21} = r_4 + r_{19}$, equation (189) together with the corresponding error bounds (191) are immediate consequences.

To ensure existence of the limit of $r_{20}(x, y, \epsilon u)$ as $\epsilon \downarrow 0$ for $x, y \in \Sigma_t$ with $x \neq y$, we use the existence of the limits $\lim_{\epsilon \downarrow 0} r_3(x, y, \epsilon t)$ and $\lim_{\epsilon \downarrow 0} r_{18}(x, y, \epsilon u)$. The existence of the former limit was proven in Lemma 2.13, and existence of the latter limit follows by the same argument, i.e., from the fact that the functions D and $\partial_\mu D$ are continuous at z , and that r_{18} is explicitly given in terms of D and its derivative. This yields the claim. \square

Corollary 2.16. *The error terms $r_{20}(\cdot, \cdot, \epsilon u)$ and $r_{21}(\cdot, \cdot, \epsilon u)$ in (189) give rise to bounded linear operators $R_{20}^{\epsilon u}(t), R_{21}^{\epsilon u}(t) : \mathcal{H}_{\Sigma_t} \hookrightarrow \mathcal{H}_{\Sigma_t}$ with matrix elements*

$$\langle \phi, R_{20}^{\epsilon u}(t) \psi \rangle = \int_{x \in \Sigma_t} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4 x) r_{20}(x, y, \epsilon u) i_\gamma(d^4 y) \psi(y), \quad \psi, \phi \in \mathcal{H}_{\Sigma_t} \quad (217)$$

and similarly for $r_{21}(x, y, \epsilon u)$. They fulfill:

- (i) *The operators $R_{20}^{\epsilon u}(t)$, $\epsilon \geq 0$, are Hilbert-Schmidt operators. There is a constant $C_{13}(A, u, \Sigma)$ such that $\sup_{t \in T, \epsilon > 0} \|R_{20}^{\epsilon u}(t)\|_{I_2(\mathcal{H}_{\Sigma_t})} \leq C_{13}$. Furthermore,*

$$\lim_{\epsilon \downarrow 0} \|R_{20}^{\epsilon u}(t) - R_{20}^0(t)\|_{I_2(\mathcal{H}_{\Sigma_t})} = 0. \quad (218)$$

- (ii) $\sup_{t \in T} \|R_{21}^{\epsilon u}(t)\|_{\mathcal{H}_{\Sigma_t} \hookrightarrow \mathcal{H}_{\Sigma_t}} \leq O_{A, u, \Sigma}(\sqrt{\epsilon})$.

Proof. (i) For $\psi, \phi \in \mathcal{H}_{\Sigma_t}$, using the bound (191) for r_{20} , we find uniformly for $\epsilon > 0$ and $t \in T$ that

$$\|R_{20}^{\epsilon u}(t)\|_{I_2(\mathcal{H}_{\Sigma_t})}^2 = \int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} \text{trace} [\gamma^0 r_{20}(x, y, \epsilon u)^* \gamma^0 \Gamma(\mathbf{x}) r_{20}(x, y, \epsilon u) \Gamma(\mathbf{y})] d^3 \mathbf{y} d^3 \mathbf{x} \quad (219)$$

$$\leq C_{14} \int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} \left[\frac{e^{-C_D |\mathbf{y} - \mathbf{x}|}}{|\mathbf{y} - \mathbf{x}|} \right]^2 (1_K(x) \vee 1_K(y)) d^3 \mathbf{y} d^3 \mathbf{x} < \infty. \quad (220)$$

for some constant $C_{14}(A, u, \Sigma)$. The limit $R_{20}{}^{\epsilon u}(t) \xrightarrow{\epsilon \downarrow 0} R_{20}{}^0(t)$ in the $I_2(\mathcal{H}_{\Sigma_t})$ norm is implied by the point-wise convergence (192) stated in Lemma 2.15 and the point-wise bound (191), using dominated convergence.

(ii) For $\psi, \phi \in \mathcal{H}_{\Sigma_t}$, using the bound in (191) for r_{21} and the Cauchy-Schwarz inequality, we find analogously to the calculation (92)–(95):

$$|\langle \phi, R_{21}{}^{\epsilon u}(t) \psi \rangle| \leq O_{A,u,\Sigma}(\sqrt{\epsilon}) \int_{\mathbf{x} \in \mathbb{R}^3} \int_{\mathbf{y} \in \mathbb{R}^3} |\phi(\mathbf{x})| |\Gamma(\mathbf{x})| d^3 \mathbf{x} \frac{e^{-C_D |\mathbf{y} - \mathbf{x}|}}{|\mathbf{y} - \mathbf{x}|^{5/2}} |\Gamma(\mathbf{y})| d^3 \mathbf{y} |\psi(\mathbf{y})| \quad (221)$$

$$\leq O_{A,u,\Sigma}(\sqrt{\epsilon}) \int_{\mathbf{z} \in \mathbb{R}^3} \frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^{5/2}} d^3 \mathbf{z} \|\phi\| \|\psi\|, \quad (222)$$

which is finite and uniform in t .

The existence of the bounded linear operators $R_{20}{}^{\epsilon u}(t), R_{21}{}^{\epsilon u}(t) : \mathcal{H}_{\Sigma_t} \hookrightarrow \mathcal{H}_{\Sigma_t}$ follows. \square

Finally, we prove the Theorem 2.8 with the collected ingredients.

Proof of Theorem 2.8. With justifications given below, we find that for $\phi, \psi \in \mathcal{C}_A$

$$\left\langle \phi|_{\Sigma_{t_1}}, (P_{\Sigma_{t_1}}^A + S_{\Sigma_{t_1}}^A) \psi|_{\Sigma_{t_1}} \right\rangle - \left\langle \phi|_{\Sigma_{t_0}}, (P_{\Sigma_{t_0}}^A + S_{\Sigma_{t_0}}^A) \psi|_{\Sigma_{t_0}} \right\rangle \quad (223)$$

$$= \lim_{\epsilon \downarrow 0} \left(\int_{x \in \Sigma_{t_1}} \int_{y \in \Sigma_{t_1}} - \int_{x \in \Sigma_{t_0}} \int_{y \in \Sigma_{t_0}} \right) \overline{\phi(x)} i_\gamma(d^4 x) (p^{A,\epsilon u} + s_{\Sigma_t}^{A,\epsilon u})(x, y) i_\gamma(d^4 y) \psi(y) \quad (224)$$

$$= \lim_{\epsilon \downarrow 0} \int_{t_0}^{t_1} \int_{x \in \Sigma_t} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4 x) \left[-i \mathcal{D}_t^A (p^{A,\epsilon u} + s_{\Sigma_t}^{A,\epsilon u}) + \frac{\partial s_{\Sigma_t}^{A,\epsilon u}}{\partial t} \right] (x, y) i_\gamma(d^4 y) \psi(y) dt \quad (225)$$

$$= \lim_{\epsilon \downarrow 0} \int_{t_0}^{t_1} \int_{x \in \Sigma_t} \int_{y \in \Sigma_t} \overline{\phi(x)} i_\gamma(d^4 x) \left[-i r_{20}(x, y, \epsilon u) - i r_{21}(x, y, \epsilon u) + \frac{\partial s_{\Sigma_t}^{A,\epsilon u}}{\partial t}(x, y) \right] \cdot i_\gamma(d^4 y) \psi(y) dt \quad (226)$$

$$= \lim_{\epsilon \downarrow 0} \int_{t_0}^{t_1} \left[-i \langle \phi|_{\Sigma_t}, R_{20}{}^{\epsilon u}(t) \psi|_{\Sigma_t} \rangle - i \langle \phi|_{\Sigma_t}, R_{21}{}^{\epsilon u}(t) \psi|_{\Sigma_t} \rangle + \left\langle \phi|_{\Sigma_t}, \dot{S}_{\Sigma_t}^{A,\epsilon u}, \psi|_{\Sigma_t} \right\rangle \right] dt \quad (227)$$

In the first step from (223) to (224) we expressed the matrix elements of the operators $P_\Sigma^{\lambda A}$ and S_Σ^A in terms of the respective integral kernels $p^{\epsilon u, \lambda A}$ and $s_\Sigma^{\epsilon u, A}$ given in Lemma 2.3 and part (i) of Lemma 2.7. The step from (224) to (225) follows from Corollary 2.12. The step from (225) to (226) is a consequence of equation (189) in Lemma 2.15. Finally, in the step from (226) to (227) we have used that the integral kernels $r_{20}(\cdot, \cdot, \epsilon u)$, $r_{21}(\cdot, \cdot, \epsilon u)$, and $\partial s_{\Sigma_t}^{A,\epsilon u} / \partial t$ give rise to bounded operators $R_{20}{}^{\epsilon u}(t)$, $R_{21}{}^{\epsilon u}(t)$, and $\dot{S}_{\Sigma_t}^{A,\epsilon u}$ as ensured by Corollary 2.16 and part (ii) of Lemma 2.7.

Claim (ii) of Corollary 2.16 implies that $R_{21}{}^{\epsilon u}(t)$ converges to zero in operator norm as $\epsilon \downarrow 0$, uniformly in $t \in T$. Furthermore, claim (i) of Corollary 2.16 and part (ii) of Lemma 2.7 guarantee that $-i R_{20}{}^{\epsilon u}(t) + \dot{S}_{\Sigma_t}^{A,\epsilon u}$ converges in the $I_2(\mathcal{H}_{\Sigma_t})$ norm to a Hilbert-Schmidt operator $R(t) := -i R_{20}{}^0(t) + \dot{S}_{\Sigma_t}^{A,0}$ such that $\sup_{t \in T} \|R(t)\|_{I_2(\mathcal{H}_{\Sigma_t})} < \infty$. Calculation

(223)–(227) can now be rewritten in the form of claim (47):

$$\left\langle \phi|_{\Sigma_{t_1}}, (P_{\Sigma_{t_1}}^A + S_{\Sigma_{t_1}}^A) \psi|_{\Sigma_{t_1}} \right\rangle - \left\langle \phi|_{\Sigma_{t_0}}, (P_{\Sigma_{t_0}}^A + S_{\Sigma_{t_0}}^A) \psi|_{\Sigma_{t_0}} \right\rangle = \int_{t_0}^{t_1} \langle \phi|_{\Sigma_t}, R(t) \psi|_{\Sigma_t} \rangle dt \quad (228)$$

at first for $\phi, \psi \in \mathcal{C}_A$, but then extended by a density argument to $\phi, \psi \in \mathcal{H}_A$. Since the operators $U_{A\Sigma}$ are unitary, we get the estimate

$$\left\| U_{A\Sigma_{t_1}} (P_{\Sigma_{t_1}}^A + S_{\Sigma_{t_1}}^A) U_{\Sigma_{t_1}A} - U_{A\Sigma_{t_0}} (P_{\Sigma_{t_0}}^A + S_{\Sigma_{t_0}}^A) U_{\Sigma_{t_0}A} \right\|_{I_2(\mathcal{H}_A)} \quad (229)$$

$$\leq \int_{t_0}^{t_1} \|R(t)\|_{I_2(\mathcal{H}_{\Sigma_t})} dt < \infty. \quad (230)$$

This proves the claim. \square

Proof of Theorem 2.5. As a consequence of Theorem 2.8 and Lemma 2.7 claim (41) holds for the special case $\lambda = \lambda^A$. For general $\lambda \in \mathcal{G}(A)$, Theorem 2.4 implies $P_\Sigma^A - P_\Sigma^\lambda \in I_2(\mathcal{H}_\Sigma)$ which concludes the proof for the general case. \square

A Appendix

In this appendix we provide auxiliary estimates for the covariant functions D , its derivatives, and p^- needed in the proofs of the main results.

Lemma A.1 (Upper bounds). *Let u be a time-like four-vector. For all space-like $z \in \mathbb{R}^4$ with $|z^0| \leq V_{\max} |\mathbf{z}|$ and $\epsilon \geq 0$ with $w = z + i\epsilon u \neq 0$ we have the following bounds with the constant $C_D(V_{\max}) = \frac{m}{2} \sqrt{1 - V_{\max}^2}$, reading $1/0$ as $+\infty$:*

$$|w^\mu w^\nu D(w)| \leq O_{u, V_{\max}} (e^{-C_D |\mathbf{z}|}), \quad (231)$$

$$|D(w)| \leq O_{V_{\max}} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^2} \right), \quad (232)$$

$$|r(w)^2 \partial_\mu D(w)| \leq O_{u, V_{\max}} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|} \right), \quad (233)$$

$$|\partial_\mu D(w)| \leq O_{u, V_{\max}} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^3 \vee \epsilon^3} \right), \quad (234)$$

$$|w^\nu \partial_\mu D(w)| \leq O_{u, V_{\max}} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^2 \vee \epsilon^2} \right), \quad (235)$$

$$|\epsilon u^\mu \partial_\nu D(w)| \leq O_{u, V_{\max}} \left(\frac{\sqrt{\epsilon} e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^{5/2}} \right), \quad (236)$$

$$|\partial_\nu [r(w)^2 \partial_\mu D(w)]| \leq O_{u, V_{\max}} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^2} \right), \quad (237)$$

$$\|p^-(w)\| \leq O_{u, V_{\max}} \left(\frac{e^{-C_D |\mathbf{z}|}}{|\mathbf{z}|^3} \right). \quad (238)$$

For $\epsilon = 0$ one may take, e.g., $u = (-1, 0, 0, 0)$. In this case the u -dependence of the constants in (231)-(238) drops out.

Lemma A.2 (Lower Bound). *For all space-like $z \in \mathbb{R}^4 \setminus \{0\}$ one has the lower bound*

$$\|p^-(z)\| \geq C_{15} \frac{e^{-m|z|}}{|z|^3} \quad (239)$$

with a positive numerical constant C_{15} .

The proofs have been carried out in [4]. However, they can also be inferred from the asymptotic behavior of the modified Bessel function K_1 and its derivative given in [1, Chapters 9.6 and 9.7].

Acknowledgment This work was partially funded by the Elite Network of Bavaria through the Junior Research Group “Interaction between Light and Matter”.

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